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SYNTHESIS OF THREE-TERMINAL $\pm R, C$ NETWORKS

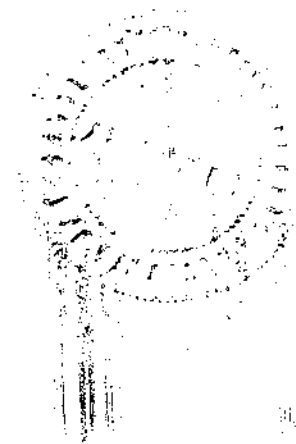
A THESIS

Presented to
the Faculty of the Graduate Division
by
Charles Lamar Phillips

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in the School
of Electrical Engineering

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SYNTHESIS OF THREE-TERMINAL $\pm R, C$ NETWORKS

Approved:

Date approved by Chairman: Dec. 13, 1962

DEDICATION

This thesis is dedicated to my father, Joe B. Phillips.

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SUMMARY

In this investigation, a general method is developed which will synthesize a two-port network when all three short-circuit admittance functions, Y_{11} , Y_{12} , and Y_{22} , which are realizable by using positive resistors, capacitors, and negative resistors, are prescribed. The twoport is a three-terminal network. Networks containing positive resistors, capacitors, and negative resistors are referred to as $\pm R, C$ networks.

The method of synthesis involves the development of a separate component network for each type of pole that is realizable by a $\pm R, C$ network. The short-circuit admittance functions are then realized by the parallel connection of the required component networks. The parallel connection has the advantage that some of the negative resistors appearing in the different component networks may be in parallel in the final connection, and thus the number of required negative resistors is reduced. All negative resistors are placed across either the input or the output terminals or the top terminals of the component networks whenever possible. It is shown that all the required negative resistors cannot be placed in these positions for every type of pole. Further, it is shown that the number of negative resistors required to synthesize a compact set of admittance functions is equal to the number of poles not on the finite negative real axis plus at most three. The synthesis technique will synthesize any set of short-circuit admittance functions that are realizable by $\pm R, C$ networks.

The basic synthesis technique developed will not necessarily require a minimum number of negative resistors. Under certain conditions, the combination of two poles is realizable without the use of negative resistors, or with the use of fewer negative resistors than the separate synthesis of the poles requires. All possible pole combinations are investigated. The conditions are determined for which the synthesis of the combinations of poles requires fewer negative resistors than the separate synthesis of the poles.

The use of transformers to reduce the number of required negative resistors is also investigated. It is found that, in general, there is a saving of one negative resistor if one transformer is used, as compared to the case that no transformer is used. Furthermore, in general, there is a saving of two negative resistors if the number of transformers used is not limited, as compared to the case that no transformer is used. However, for one special case, four negative resistors may be saved by the unlimited use of transformers. In general, the number of transformers required by this method is one plus the number of poles in the admittance functions.

Stability criteria for $\pm R, C$ two-port networks are developed in terms of the pole-zero locations of the short-circuit admittance functions for certain specified source admittances and load admittances. The source and load admittances considered are purely conductive admittances, zero admittances, and infinite admittances. It is found that in some cases, y_{11} and y_{22} may have one or two zeros and/or one or two poles on the positive real axis with the resulting network stable.

The most lenient requirements for y_{11} and y_{22} occur for the case that both the source admittance and the load admittance are finite conductances. For this case, y_{11} and y_{22} may have two zeros and two compact poles on the positive real axis with the resulting network stable.

The necessary and sufficient conditions for three different voltage transfer functions to be realizable by a stable $\pm R, C$ network are derived. The transfer functions are the voltage transfer function for the open-circuited network, the voltage transfer function for the singly-terminated network, and the voltage transfer function for the double-loaded network. The necessary and sufficient conditions were found to be identical for these three transfer functions. These conditions are:

1. The coefficients in the numerator polynomial are real.
2. The zeros of the denominator polynomial occur only on the negative real axis.
3. The degree of the numerator polynomial is not more than one greater than that of the denominator polynomial.

The synthesis technique is applied to the realization of these transfer functions. The realization of the voltage transfer functions for either the open-circuited network or the singly-loaded network usually requires two negative resistors. Procedures are developed by which the number of negative resistors required for the realization of either of these transfer functions can sometimes be reduced to less than two.

The realization of the voltage transfer function for the double-loaded network requires at most three negative resistors when the degree of the numerator is less than or equal to the degree of the denominator of the transfer function. A maximum of five negative resistors is required if the degree of the numerator is one greater than the degree of the denominator. Conditions are derived such that the number of negative resistors required can be determined from the values of the source and the load resistances and from the degree of the numerator as compared to the degree of the denominator.

Numerical examples are included to illustrate all results given.

CHAPTER I

INTRODUCTION

Recent developments in certain fields of electrical engineering, notably in semiconductors, low-temperature devices, and feedback amplifiers, have led to an interest in the synthesis of active networks. Active network synthesis refers to the synthesis of a class of networks that are lumped, linear, and finite, but not passive and not necessarily bilateral. Many active network synthesis techniques (1) have been developed using controlled sources, negative impedance converters, gyrators, and negative resistors. Most of the synthesis procedures using controlled sources, negative impedance converters, and gyrators apply to active RC networks. In these procedures, positive resistors and capacitors are used in conjunction with one of the types of active elements mentioned.

Some work has been done in the active synthesis field using negative resistors as active elements. Carlin and Youla (2) considered the use of negative resistors with the lumped passive elements (RLC) and derived the realizability conditions of a real rational $n \times n$ immittance matrix. Youla and Smilen (3) developed a synthesis method for the realization of single-stage broadband negative-resistance amplifiers using tunnel diodes. Sard (4) made an analysis of certain connections of negative-resistance amplifiers operated with lumped-constant filters to give an overall maximally flat amplitude response. Kawakami, Yanagisawa,

and Shibayama (5) developed a design procedure for highly selective bandpass filters that use negative resistors. Kinariwala (6) developed the realizability conditions for a network consisting of a tunnel diode which is embedded in an otherwise lossless and reciprocal network. Weinberg (7) developed synthesis techniques for realizing prescribed frequency characteristics by amplifier networks that contain tunnel diodes or masers, with some of the techniques based on uniform and nonuniform reverse predistortion.

All of these synthesis techniques use inductances in addition to positive resistors, capacitors, and negative resistors. The problem of networks containing negative resistors, positive resistors, and capacitors (this class of networks will be referred to as $\pm R, C$ networks) has been investigated only sporadically. Kinariwala (8) and Bello (9) independently derived the necessary and sufficient conditions for the existence of $\pm R, C$ networks. Su (10) used negative resistors with RC networks in a technique for developing any desired RC driving-point immittance to produce right-halfplane transmission zeros.

Allied to the synthesis techniques of $\pm R, C$ networks are the synthesis techniques of RC networks. Lucal (11) investigated the problem of the realization of RC short-circuit admittance functions that are not symmetric. He was not able to rigorously establish the necessary and sufficient conditions for realizability. However, he was able to develop a synthesis technique with which a majority of functions that meet certain specified conditions can be synthesized.

The $\pm R, C$ functions offer properties not possessed by RC functions. As was shown by Su (10), negative resistors used in conjunction with RC networks can produce transmission zeros in the right halfplane. Also, the constant multiplier of y_{12} can be specified beforehand. In fact, there is no bound on this constant. And there are fewer restrictions on the specifications of y_{11} and y_{22} . Under certain conditions, y_{11} and y_{22} can have two zeros and two poles on the positive real axis, and the network still may be stable.

The purpose of this investigation is the development of a general synthesis technique for two-port networks when all three short-circuit admittance functions, y_{11} , y_{12} , and y_{22} , which are realizable by using positive resistors, capacitors, and negative resistors, are prescribed. Stability criteria are investigated for this class of networks. The application of the synthesis technique to various transfer functions is also investigated.

CHAPTER II

GENERAL SYNTHESIS TECHNIQUE

The necessary and sufficient conditions for the physical realizability of $\pm R, C$ networks were derived independently by Kinariwala (8) and Bello (9). These conditions were stated by Kinariwala in the following theorem:

Necessary and sufficient conditions for the physical realizability of $\pm R, C$ n -ports are that the elements of the open-circuit impedance matrices have poles that are simple and are restricted to the real axis in the complex frequency plane. The matrices of the residues in all of the finite poles must be positive semidefinite, and those in the poles at infinity must be negative semidefinite.

In terms of the short-circuit admittance functions of a two-port network, the theorem states that these functions can be expressed as

$$\begin{aligned} y_{11} &= g_{11} + k_{11\infty}s + \frac{k_{110}}{s} + \sum \frac{k_{11i}s}{s + s_i} \\ y_{12} &= g_{12} + k_{12\infty}s + \frac{k_{120}}{s} + \sum \frac{k_{12i}s}{s + s_i} \\ y_{22} &= g_{22} + k_{22\infty}s + \frac{k_{220}}{s} + \sum \frac{k_{22i}s}{s + s_i} \end{aligned} \quad (1)$$

and that the necessary and sufficient conditions for these conditions to be realizable by a $\pm R, C$ network are:

- a. All finite poles must fall on the real axis.
- b. The residues of y_{11}/s and y_{22}/s in all poles on the

negative real axis are positive and real, the residues of y_{11}/s and y_{22}/s in all poles on the positive real axis are negative and real, and the residues of y_{12}/s in all poles are real.

c. $k_{11\infty}$ and $k_{22\infty}$ are real and nonnegative, and k_{110} and k_{220} are real and nonpositive.

d. In all poles, $k_{11} k_{22} - k_{12}^2 \geq 0$.

e. g_{11} , g_{12} , and g_{22} are real.

The short-circuit admittance functions are defined with the two-port currents having the directions shown in Figure 1.

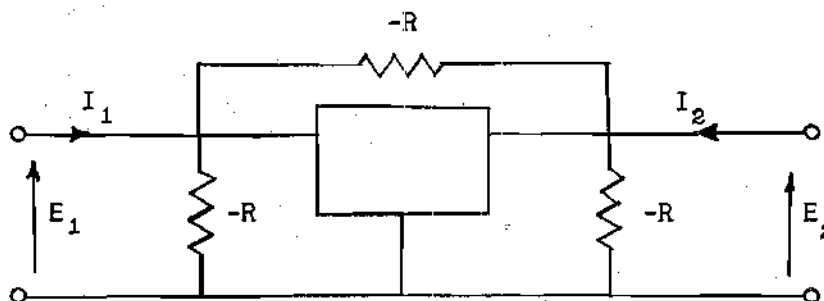


Figure 1. Desirable Locations for Negative Resistors

The method of synthesis devised in this investigation is a parallel ladder development. A network is developed that will realize each type of pole. Consider a typical pole, and let s_i be positive.

$$y_{11} = \frac{k_{11i}s}{s + s_i}$$

$$y_{12} = \frac{k_{12i}s}{s + s_i}$$

$$y_{22} = \frac{k_{22i}s}{s + s_i}$$

Lucal (11) has shown that this set of admittance functions cannot be realized by a three-terminal RC network unless

$$a. \quad k_{12i} \leq 0,$$

$$b. \quad k_{11i} \geq |k_{12i}|; \quad k_{22i} \geq |k_{12i}|.$$

For the case of a compact pole ($k_{11i} k_{22i} - k_{12i}^2 = 0$, $k_{12i} \neq 0$), where $k_{11i} \neq |k_{12i}|$, the pole cannot be realized as an RC three-terminal network. However, as is shown in the next section, this pole can be realized as a $\pm R, C$ network using one negative resistor.

Since the networks realizing the poles are to be connected in parallel, it would be desirable to have the negative resistors in the networks located such that as many of these negative resistors as possible are in parallel and can be combined. If the negative resistors are located in any of the three locations shown in Figure 1, the possibility of parallel connections exists.

The component networks will be given in the order that the functions appear in (1). Consider first the conductance terms.

$$y_{11} = g_{11}$$

$$y_{12} = g_{12} \tag{2}$$

$$y_{22} = g_{22}$$

These functions are realizable by the network shown in Figure 2. It is possible for all the resistors to be negative for a given set of

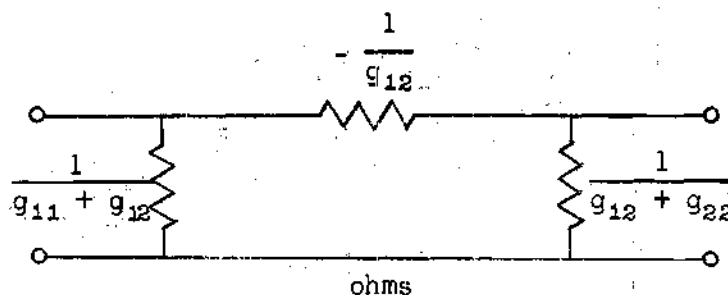


Figure 2. Network for Realizing Conductance Terms

admittances. All of the resistors that may be negative are in the desirable locations.

Consider next the pole at infinity.

$$\begin{aligned} y_{11} &= k_{11}s \\ y_{12} &= k_{12}s \\ y_{22} &= k_{22}s \end{aligned} \tag{3}$$

If the pole is compact, it can be realized by the network of Figure 3. If the pole is not compact, sufficient admittance should be subtracted from either y_{11} or y_{22} to make the pole compact. The subtracted admittance can be realized by a capacitor in parallel with either the input or the output of the network of Figure 3. The value of R in Figure 3 is arbitrary and should be chosen negative. This choice makes the resistor

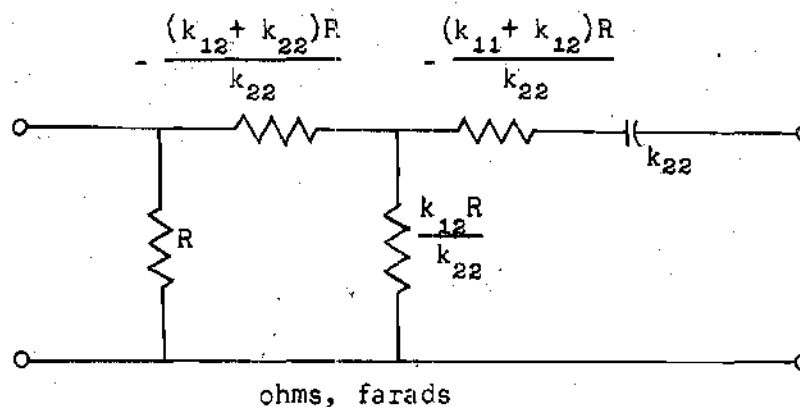


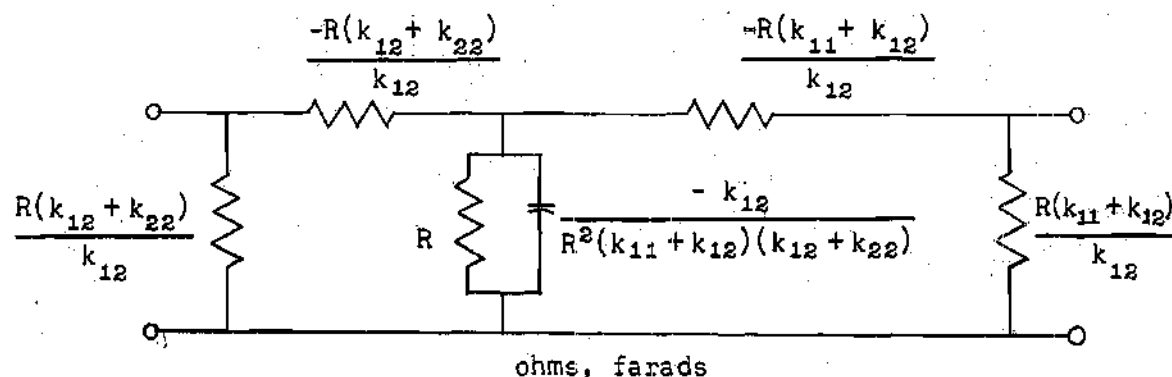
Figure 3. Network for Realizing the Pole at Infinity

across the input negative. Also, one of the other three resistors will be negative with the rest positive. Thus, one negative resistor will be internal to the network, and cannot be combined with other negative resistors. If R is chosen positive, two negative resistors will be internal to the network. Negative resistors which are not in the desirable locations shown in Figure 1 will be referred to an "internal negative resistors."

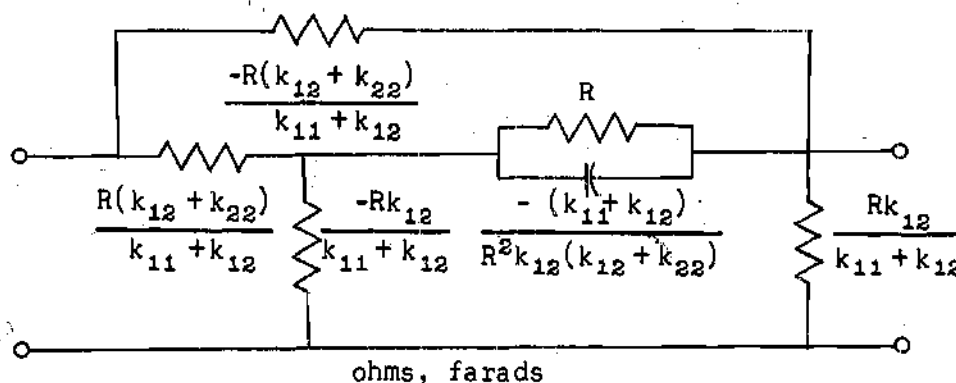
Consider next the pole at the origin.

$$\begin{aligned} Y_{11} &= \frac{k_{11}}{s} \\ Y_{12} &= \frac{k_{12}}{s} \\ Y_{22} &= \frac{k_{22}}{s} \end{aligned} \quad (4)$$

The compact pole can be realized by either the network of Figure 4a or that of Figure 4b. Two different networks are used because, under certain



(a)



(b)

Figure 4. Network for Realizing the Pole at the Origin

conditions, one of the two networks will require two internal negative resistors. For k_{12} negative, the network of Figure 4a is used with R arbitrary but negative. Then the resistors across the input and the output are negative, in addition to the resistor in parallel with the capacitor. For k_{12} positive and $|k_{11}| > k_{12}$, the network of Figure 4b is used with R arbitrary but negative. Then the resistors across the top terminals and the output are negative, in addition to the

resistor in parallel with the capacitor. For k_{12} positive and $|k_{11}| < k_{12}$, the network of Figure 4b is used with R arbitrary and negative, but with the input and output ports reversed, and with the constants k_{11} and k_{22} interchanged.

Consider a pole on the negative real axis.

$$\begin{aligned} Y_{11} &= \frac{k_{11}s}{s + s_i} \\ Y_{12} &= \frac{k_{12}s}{s + s_i} \\ Y_{22} &= \frac{k_{22}s}{s + s_i} \end{aligned} \quad (5)$$

The compact pole can be realized by the network of either Figure 5a or Figure 5b. If k_{12} is negative and $k_{11} < |k_{12}|$, the network of

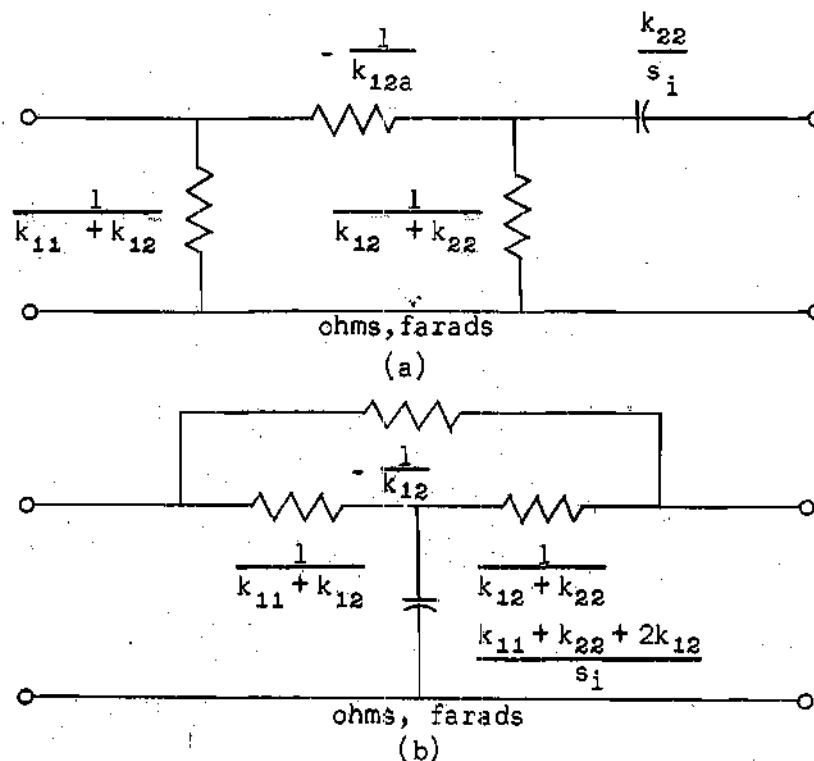


Figure 5. Networks for Realizing a Pole on the Real Axis

Figure 5a is used. Then the only negative resistor required is across the input. If k_{12} is negative and $k_{11} > |k_{12}|$, the network of Figure 5a is used, with the input and output ports reversed and the constants k_{11} and k_{22} interchanged. Then the only negative resistor appears across the output. If k_{12} is positive, the network of Figure 5b is used. Then the only negative resistor required appears across the top of the network.

Consider a pole on the positive real axis. The form of the pole is the same as that for a pole on the negative real axis. However, k_{11} and k_{22} are negative. If the pole is compact, it can be realized by either the network of Figure 5a or that of Figure 5b. If k_{12} is negative, the network of Figure 5a is used. Then the input resistor and one of the other two resistors are negative. If k_{12} is positive, the network of Figure 5b should be used. Then the resistor across the top terminals and one of the other two resistors are negative.

An examination of the component networks given shows that, in general, in addition to the negative resistors shown in Figure 1, one negative resistor will be required for the pole at infinity, one for the pole at zero, and one for each pole on the positive real axis. Or, the number of negative resistors required to synthesize a compact set of admittance functions is equal to the number of poles not on the finite negative real axis plus at most three. The number of poles allowable on the positive real axis is determined by stability considerations, and this question is investigated in Chapter IV.

In general, the number of negative resistors required to synthesize a set of admittance functions could be reduced if the internal

negative resistors required could be eliminated. Internal negative resistors are required for poles not on the finite negative real axis. However, as is shown in Appendix I, synthesizing networks for these poles that do not include internal negative resistors are not possible.

CHAPTER III

REDUCTION OF THE NUMBER OF REQUIRED NEGATIVE RESISTORS

Straightforward application of the general synthesis method presented in Chapter II does not necessarily give a minimum number of negative resistors for a given set of admittances. Two methods for reducing the number of required negative resistors are developed in this chapter. The first method utilizes the combination of two poles. The synthesis of the combination of the two poles, rather than the synthesis of each pole separately, will lead to a reduction in the number of required negative resistors under certain conditions. These conditions are developed in this chapter. The second method utilizes transformers to reduce the number of required negative resistors. In this method, the use of an unlimited number of transformers is first considered, and then the use of only one transformer is considered.

Simultaneous Synthesis of Two Poles

Consider a set of admittance functions made up of two compact poles which occur on the finite real axis.

$$\begin{aligned}
 y_{11} &= \frac{k_{11a}s}{s+s_a} + \frac{k_{11b}s}{s+s_b} = \frac{s^2(k_{11a}+k_{11b}) + s(k_{11a}s_b + k_{11b}s_a)}{s^2 + s(s_a + s_b) + s_a s_b} \\
 y_{12} &= \frac{k_{12a}s}{s+s_a} + \frac{k_{12b}s}{s+s_b} = \frac{s^2(k_{12a}+k_{12b}) + s(k_{12a}s_b + k_{12b}s_a)}{s^2 + s(s_a + s_b) + s_a s_b} \\
 y_{22} &= \frac{k_{22a}s}{s+s_a} + \frac{k_{22b}s}{s+s_b} = \frac{s^2(k_{22a}+k_{22b}) + s(k_{22a}s_b + k_{22b}s_a)}{s^2 + s(s_a + s_b) + s_a s_b}
 \end{aligned} \tag{6}$$

The set of open-circuit impedance functions for this set of admittances is:

$$\begin{aligned} z_{11} &= \frac{y_{22}}{|y|} = \frac{1}{k} \left[(k_{22a} + k_{22b}) + \frac{k_{22a}s_b + k_{22b}s_a}{s} \right] \\ z_{12} &= \frac{-y_{12}}{|y|} = -\frac{1}{k} \left[(k_{12a} + k_{12b}) + \frac{k_{12a}s_b + k_{12b}s_a}{s} \right] \\ z_{22} &= \frac{y_{11}}{|y|} = \frac{1}{k} \left[(k_{11a} + k_{11b}) + \frac{k_{11a}s_b + k_{11b}s_a}{s} \right] \end{aligned} \quad (7)$$

where

$$k = k_{11a}k_{22b} + k_{11b}k_{22a} - 2k_{12a}k_{12b}$$

This set of impedances is realizable by a $\pm R, C$ tee network if the residues of z_{11} , z_{12} , and z_{22} in the pole at zero are positive and if the residues of z_{11} and z_{22} in this pole are each greater than or equal to the residue of z_{12} . Seven cases have led to useful results.

These are:

- Case A: $s_a > 0$, $s_b > 0$, $k_{12a} < 0$, $k_{12b} < 0$.
- Case B: $s_a > 0$, $s_b > 0$, $k_{12a} > 0$, $k_{12b} < 0$.
- Case C: $s_a < 0$, $s_b > 0$, $k_{12a} < 0$, $k_{12b} < 0$.
- Case D: $s_a < 0$, $s_b > 0$, $k_{12a} > 0$, $k_{12b} < 0$.
- Case E: $s_a < 0$, $s_b > 0$, $k_{12a} > 0$, $k_{12b} > 0$.
- Case F: $s_a < 0$, $s_b < 0$, $k_{12a} > 0$, $k_{12b} > 0$.
- Case G: $s_a < 0$, $s_b < 0$, $k_{12a} > 0$, $k_{12b} < 0$.

All other possible cases have been investigated and have been found to produce nothing useful. These other cases are presented in Appendix II.

Case A

For this case, $s_a > 0$, $s_b > 0$, $k_{12a} < 0$, and $k_{12b} < 0$.

From (7), it is seen that the constant term of z_{12} is always positive. If no negative resistor is to be used in the tee network, it is necessary that

$$\begin{aligned} k_{11a} + k_{11b} &\geq - (k_{12a} + k_{12b}) \\ k_{22a} + k_{22b} &\geq - (k_{12a} + k_{12b}) \end{aligned} \quad (8)$$

This requires that the residue in y_{11}/s in one pole be less and the residue in the other pole be greater, in magnitude, than those of y_{12}/s in the respective poles. Under these conditions, the synthesis of the two poles separately would require two negative resistors.

If the residues of y_{11}/s are both greater or are both less, in magnitude, than those of y_{12}/s in the respective poles, then the separate synthesis requires one negative resistor. Since the synthesis of the combination also requires one negative resistor, there is no advantage in combining the poles.

For the capacitive terms to be realizable as a tee network, it is necessary that

$$\begin{aligned} k_{11a}s_b + k_{11b}s_a &\geq - (k_{12a}s_b + k_{12b}s_a) \\ k_{22a}s_b + k_{22b}s_a &\geq - (k_{12a}s_b + k_{12b}s_a) \end{aligned} \quad (9)$$

If (8) and (9) are satisfied, the functions can be synthesized using no negative resistors. If (9) is satisfied, and if only one of (8) is satisfied, the synthesis of the combination requires only one negative resistor.

If neither of (8) is satisfied, or if (9) is not satisfied, then it may be possible to combine part of one pole with the other pole to form a set of functions realizable as an RC network. The remaining part of the pole can be realized using one negative resistor and, thus, one negative resistor is saved.

Suppose that pole a is divided into two parts.

$$y_{11a} = \frac{rk_{11a}s}{s + s_a} + \frac{(1-r)k_{11a}s}{s + s_a}$$

$$y_{12a} = \frac{rk_{12a}s}{s + s_a} + \frac{(1-r)k_{12a}s}{s + s_a}$$

$$y_{22a} = \frac{rk_{22a}s}{s + s_a} + \frac{(1-r)k_{22a}s}{s + s_a}$$

Since the pole is compact, the breakdown must be made such that the two parts of the pole are also compact. The constant r is positive and less than one. Then, in the combination of the poles as shown in (6) and (7), k_{11a} , k_{12a} , and k_{22a} , are replaced by rk_{11a} , rk_{12a} , and rk_{22a} , respectively. Also, in the conditions given in (8) and (9), the same replacements are made. If r is chosen to satisfy (8) and (9), the following limits on r are obtained.

$$\frac{-(k_{12b} + k_{22b})}{(k_{12a} + k_{22a})} \geq r \geq \frac{-(k_{11b} + k_{12b})}{(k_{11a} + k_{12a})} \quad (10)$$

$$\frac{-s_a(k_{12b} + k_{22b})}{s_b(k_{12a} + k_{22a})} \geq r \geq \frac{-s_a(k_{11b} + k_{12b})}{s_b(k_{11a} + k_{12a})}$$

This set of relations is obtained on the assumption that $k_{11a} > |k_{12a}|$. If this inequality is not satisfied, the inequality signs in each of the above relationships must be reversed, as may be seen by noting that they are obtained by dividing by either the factor $(k_{11a} + k_{12a})$ or $(k_{12a} + k_{22a})$. One of these factors will always be negative. If this breakdown is possible, both relationships in (10) will allow r to be chosen positive and less than one. If one of the limiting values is chosen for r , one circuit element will be saved. If the constant r is found to be positive and greater than one for both relationships, a breakdown of pole b , instead of pole a , should be tried. For this breakdown to be successful, it is necessary that the two intervals for r as given in (10) be overlapping. A numerical example illustrating Case A is given in Appendix III.

Case B

For this case, $s_a > 0$, $s_b > 0$, $k_{12a} > 0$, and $k_{12b} < 0$. Two negative resistors are required in the separate synthesis of the two poles. In order that the restrictive term of z_{12} be nonnegative, $|k_{12b}| \geq k_{12a}$. Therefore, for the combination to be realizable without the use of negative resistors, $|k_{12b}| \geq k_{12a}$, $|k_{12b}s_a| \geq k_{12a}s_b$, and the conditions given in (8) and (9) must be satisfied. If $|k_{12b}s_a| \geq k_{12a}s_b$, and if only one negative resistor is required in the realization of the combination of the poles, then one negative resistor is saved.

If these conditions are not satisfied, possibly pole a can be divided into two parts in the manner described for Case A. Since both $(k_{11a} + k_{12a})$ and $(k_{12a} + k_{22a})$ are positive, the conditions stated in (10) become

$$\begin{aligned}
 r &\geq \frac{-(k_{11b} + k_{12b})}{(k_{11a} + k_{12a})} \\
 r &\geq \frac{-(k_{12b} + k_{22b})}{(k_{12a} + k_{22a})} \\
 r &\geq \frac{-s_a(k_{11b} + k_{12b})}{s_b(k_{11a} + k_{12a})} \\
 r &\geq \frac{-s_a(k_{12b} + k_{22b})}{s_b(k_{12a} + k_{22a})}
 \end{aligned} \tag{11}$$

In this set of inequalities, two of the right side values will always be negative. Therefore, it is necessary that the two values which are positive be less than one. Also, in order that the capacitive term of z_{12} be positive, it is necessary that $|k_{12b} s_a| \geq r k_{12a} s_b$.

If pole a cannot be divided successfully, then pole b can always be divided in the same manner if $r' |k_{12b} s_a| \geq k_{12a} s_b$. The constant r' , which is equal to $1/r$, is determined from (11). If r is found to be greater than one, then r' will be less than one and will satisfy the required conditions.

Case C

For this case, $s_a < 0$, $s_b > 0$, $k_{12a} < 0$, and $k_{12b} < 0$.

It is seen from Chapter II that the separate synthesis of the two poles will require two negative resistors if $|k_{11}| > |k_{12}|$ for both poles, and three negative resistors if this inequality is satisfied for one pole and not for the other. Two negative resistors are required if this inequality is not satisfied for either pole.

A realization of the combined poles requires at least one negative resistor, because of the presence of the pole on the positive real axis. Consider the expressions in (7). The constant k will always be negative, and therefore a negative resistor will always be required in the realization of z_{12} . An additional negative resistor may be required in the realization of either z_{11} or z_{22} , but not both. This follows from the fact that the difference of k_{11b} and $|k_{11a}|$ and the difference of k_{22b} and $|k_{22a}|$ cannot both be positive and greater than the sum of $|k_{12a}|$ and $|k_{12b}|$. Thus, if the capacitive terms of z_{11} , z_{12} , and z_{22} are realizable, either one or two negative resistors can be saved. For any specific poles, a substitution into (7) will show whether or not any negative resistors can be saved.

A breakdown of pole a cannot result in the saving of any negative resistors, since the remaining part of the pole requires two negative resistors. For the case in which three negative resistors are required in the separate synthesis of the poles, a breakdown of pole b may result in the saving of one negative resistor if the combined poles can be realized with only one negative resistor. The limits on r for

which this is possible are obtained by the same method as used in

Case A. These limits are:

$$\frac{-(k_{11a} + k_{12a})}{(k_{11b} + k_{12b})} \geq r \geq \frac{-(k_{12a} + k_{22a})}{(k_{12b} + k_{22b})} \quad (12)$$

$$\frac{-s_b(k_{11a} + k_{12a})}{s_a(k_{11b} + k_{12b})} \leq r \leq \frac{-s_b(k_{12a} + k_{22a})}{s_a(k_{12b} + k_{22b})}$$

These relationships are based on the assumption that $k_{11b} > |k_{12b}|$.

If this inequality is not satisfied, the sense of the inequality signs must be reversed. There is an additional condition that must be satisfied. The capacitive term of z_{12} must be positive. Or, $rk_{12b}s_a \geq |k_{12a}s_b|$.

Case D

For this case, $s_a < 0$, $s_b > 0$, $k_{12a} > 0$, and $k_{12b} < 0$.

Three negative resistors are required for the separate synthesis of these two poles.

Consider the expressions in (7). The constant k will always be negative. At least one negative resistor will always be required because of the pole on the positive real axis. The necessary conditions for three negative resistors to be required are

$$k_{11b} - |k_{11a}| > |k_{12b}| - k_{12a} > 0$$

$$k_{22b} - |k_{22a}| > |k_{12b}| - k_{12a} > 0.$$

Since k_{12a} is the geometric mean of $|k_{11a}|$ and $|k_{22a}|$, and $|k_{12b}|$ is the geometric mean of k_{11b} and k_{22b} , both conditions cannot be satisfied simultaneously. Thus, if the capacitive terms are realizable, at most two negative resistors will be required, and possibly only one.

If the combination of poles is not realizable, possibly pole b can be divided such that part of the pole in combination with pole a is realizable. For there to be a saving in negative resistors, it is necessary that the combination be realizable with only one negative resistor. The limits on r for the capacitive terms of z_{12} are the same as the second set of the inequalities in (12). For the resistive terms, the limits on r are the same as the first set of inequalities in (12), if the resistive term of z_{12} is negative, i.e., if $r|k_{12b}| > k_{12a}$. If the resistive term of z_{12} is positive, one of the limits in the first set of inequalities of (12) can be violated. These limits apply to the case that $k_{11b} > |k_{12b}|$. If $k_{11b} < |k_{12b}|$, all inequality signs should be reversed.

Case E

For this case, $s_a < 0$, $s_b > 0$, $k_{12a} > 0$, and $k_{12b} > 0$. For this combination, k of (7) is negative. The realization of the poles separately requires two negative resistors. For the combination to be worthwhile, only one negative resistor can be used. At least one negative resistor will always be required, since a pole on the positive real axis is present. For the combination to be realizable, it is necessary that the capacitive term of z_{12} be positive, or, $k_{12a}s_b \geq |k_{12b}s_a|$.

An examination of z_{12} in (7) shows that the resistive term is always positive. Thus, the combination is advantageous only if

$$-(k_{22a} + k_{22b}) \geq k_{12a} + k_{12b}$$

or

$$-(k_{11a} + k_{11b}) \geq k_{12a} + k_{12b}.$$

If the combination of poles is not directly realizable, no advantage can be taken by dividing one of the poles. For any breakdown, at least two negative resistors will be required.

Case F

For this case, $s_a < 0$, $s_b < 0$, $k_{12a} > 0$, and $k_{12b} > 0$.

The separate synthesis of the two poles requires three negative resistors.

In (7), k is always positive, and therefore the capacitive term of z_{12} is always positive. The resistive term of z_{12} is always negative and requires a negative resistor in its realization. For the combination to be worthwhile, both z_{11} and z_{22} should not require negative resistors. If the capacitive terms are realizable, the combination results in a savings of negative resistors if either

$$|k_{11a} + k_{11b}| \leq k_{12a} + k_{12b}$$

or

$$|k_{22a} + k_{22b}| \leq k_{12a} + k_{12b}.$$

Neither of these inequalities may be satisfied in any specific case, and both are never satisfied simultaneously. This is seen by adding

the two inequalities, and noting that the left side is always greater than or equal to the right side. Thus, if the capacitive terms are realizable, the synthesis of the combination of the two poles may result in a savings of one negative resistor. There is no advantage in dividing one of the poles, since the realization of the remaining part of the pole will require two negative resistors.

Case G

For this case, $s_a < 0$, $s_b < 0$, $k_{12a} > 0$, and $k_{12b} < 0$. The separate synthesis of these two poles requires four negative resistors.

In (7), k is always positive. For the capacitive term to be nonnegative, it is necessary that $|k_{12a}s_b| \geq k_{12b}s_a$. The resistive term of z_{12} may be positive or negative. However, the resistive terms of z_{11} and z_{22} are always negative. If the resistive term of z_{12} is positive, two negative resistors are required in the realization of z_{11} and z_{22} . If the resistive term of z_{12} is negative, the realization of z_{12} requires one negative resistor. The realization of z_{11} and z_{22} will require at least one additional negative resistor, since either

$$|k_{11a} + k_{11b}| > |k_{12b}| - k_{12a},$$

or

$$|k_{22a} + k_{22b}| > |k_{12b}| - k_{12a},$$

or possibly both inequalities may be satisfied. In either case, at least two negative resistors are required, and possibly three. Therefore,

if the capacitive terms are realizable, then the synthesis of the combination results in a saving of at least one negative resistor, and possibly two. There is no advantage in dividing one of the poles, since the synthesis of the remaining part of the pole will require two negative resistors.

Synthesis Using Transformers

Consider the network shown in Figure 6. For this network, y_{11} , y_{12} , and y_{22} are the admittance functions of the total network, and y'_{11} , y'_{12} , and y'_{22} are the admittance functions of the sub-network.

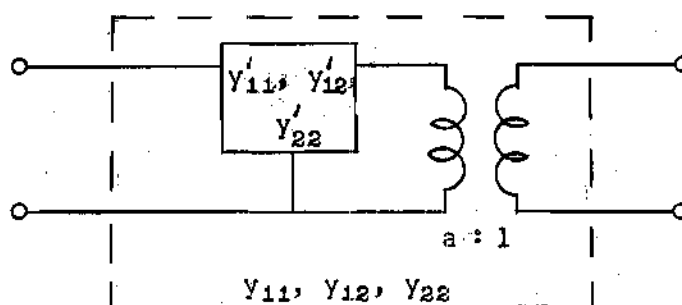


Figure 6. Synthesis Network Using a Transformer

For the network of Figure 6,

$$y_{11} = y'_{11}$$

$$y_{12} = ay'_{12}$$

$$y_{22} = a^2 y'_{22}$$

A compact pole on the negative real axis can be realized by the network of Figure 7. This network requires no negative resistors.

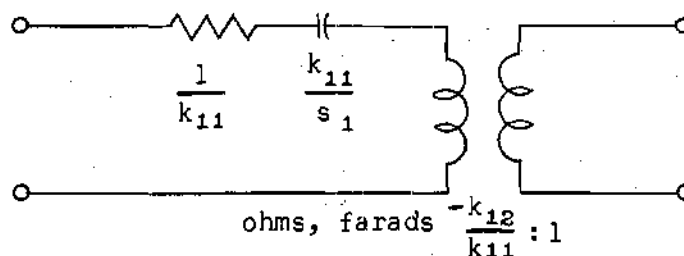


Figure 7. Network for Realizing a Pole on the Real Axis

For Figure 7,

$$Y_{11} = \frac{k_{11}s}{s + s_1}$$

$$Y_{12} = \frac{k_{12}s}{s + s_1}$$

$$Y_{22} = \frac{k_{22}s}{s + s_1}$$

If s_1 is negative for the above pole, k_{11} and k_{22} are also negative. Thus, a pole on the positive real axis also can be realized by the network of Figure 7, but the resistor will be negative. The pole at infinity can be realized by the network of Figure 7 with the resistance equal to zero and the capacitance equal to k_{11} farads.

The pole at the origin can be realized by the network of Figure 8. In this network, R is chosen to be any convenient positive value.

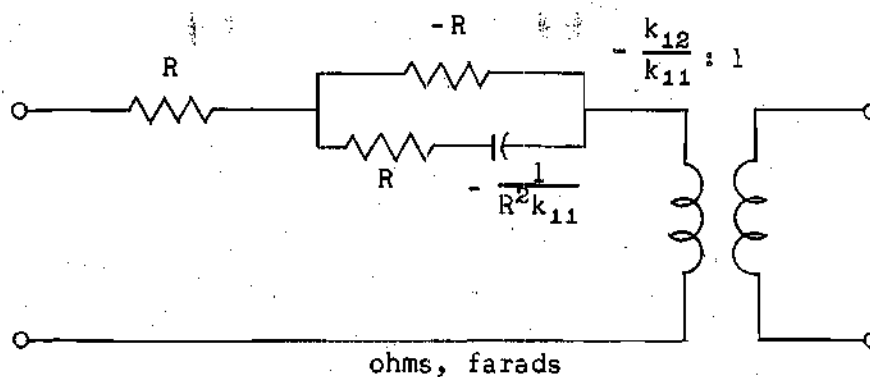


Figure 8. Network for Realizing the Pole at the Origin

For Figure 8,

$$y_{11} = \frac{k_{11}}{s}$$

$$y_{12} = \frac{k_{12}}{s}$$

$$y_{22} = \frac{k_{22}}{s}$$

To realize a set of conductance terms, the network of Figure 9 may be used.

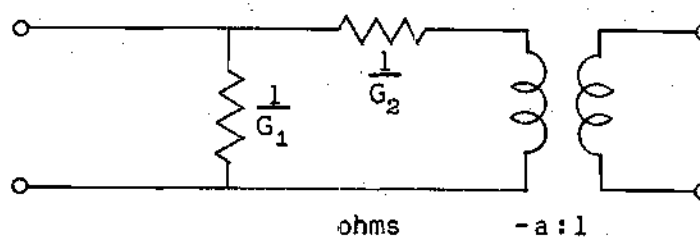


Figure 9. Network for Realizing Conductance Terms

For Figure 9,

$$y_{11} = g_{11} = G_1 + G_2$$

$$y_{12} = g_{12} = aG_2$$

$$y_{22} = g_{22} = a^2G_2$$

Thus,

$$a = \frac{g_{22}}{g_{12}}$$

$$G_1 = \frac{g_{11}g_{22} - g_{12}^2}{g_{22}}$$

$$G_2 = \frac{g_{12}^2}{g_{22}}$$

If y_{22} is zero, g_{22} in the above relationships can be assumed to be any convenient positive value, and a conductance of value $(-g_{22})$ should be connected across the output terminals to make y_{22} zero. If y_{11} and y_{22} are both negative and if $g_{11}g_{22} > g_{12}^2$, then both resistors in the network will be negative. Otherwise, at least one of the two negative resistors will be positive.

Therefore, if the number of transformers used is not restricted, any set of $\pm R, C$ short-circuit admittances can be realized with the number of negative resistors required equal to the number of finite poles not on the negative real axis, plus possibly two for the conductance terms.

If the number of transformers used is restricted to one, the network shown in Figure 10 can be used. The transformer is used to convert

the poles which have positive k_{12} 's to have negative k_{12} 's. The negative resistors shown can be combined into two negative resistors. If k_{12} is positive for the pole at infinity, the turns ratio can be chosen to realize this pole without negative resistors. If k_{12} for the pole at infinity is negative, the turns ratio can be chosen to be any convenient value. If g_{12} of the conductance terms is negative, the conductance terms can be included in the top network of Figure 10. If g_{12} is positive, the conductance terms can be included in the

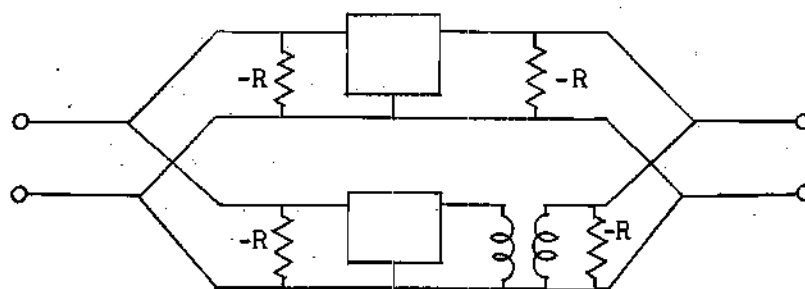


Figure 10. Synthesis Network Using One Transformer.

bottom network. In either case, no negative resistor is required across the top terminals of the network. In general, the number of negative resistors required will be two plus the number of poles not on the finite negative real axis.

In general, there is a saving of one negative resistor for the one transformer used, as compared to the case of no transformers used. Also, in general, the maximum number of negative resistors required to

synthesize a set of admittance functions is one greater in using one transformer than in using an unlimited number of transformers. The additional negative resistor is required by the pole at infinity. However, for a specific case, the number of negative resistors required may be reduced by three by the unlimited use of transformers. The additional negative resistors that may be saved are the two assigned to the conductance terms.

CHAPTER IV

STABILITY OF $\pm R, C$ NETWORKS

Since active elements are present in $\pm R, C$ networks, the question of stability naturally arises. If a natural frequency of the network falls in the right halfplane or on the $j\omega$ axis, the network is unstable. A positive real zero in y_{11} or y_{22} , or a positive real pole in y_{11} , y_{12} , and y_{22} does not necessarily mean that the network is unstable. Stability is determined by both the pole-zero locations of the admittance functions and the terminations of the network.

If every part of the network is coupled with the remainder of the network, the natural frequencies of a network can be obtained by setting to zero the impedance of any closed loop, or by setting to zero the admittance between any two nodes. If a part of the network is uncoupled from the remainder, investigation of the remainder will not reveal the natural frequencies of the uncoupled part. An example of this case is found in Case A below. To take into account the possibility of uncoupled parts appearing in the networks to be investigated, both the input circuit and the output circuit of each network will be investigated.

The stability criteria of $\pm R, C$ networks will be derived in terms of different source and load admittances and the pole-zero locations of y_{11} and y_{22} . The different source and load admittances that are to be considered are

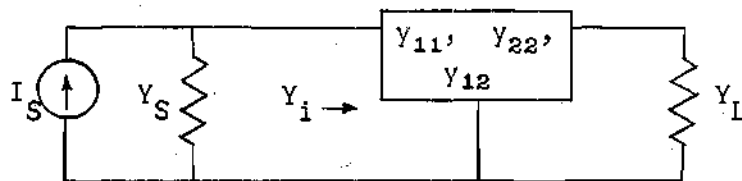


Figure 11. General Network with Terminations

$$\text{Case A: } Y_S = \infty, \quad Y_L = 0.$$

$$\text{Case B: } Y_S = \infty, \quad Y_L = G_L.$$

$$\text{Case C: } Y_S = G_S, \quad Y_L = G_L.$$

$$\text{Case D: } Y_S = 0, \quad Y_L = G_L.$$

$$\text{Case E: } Y_S = 0, \quad Y_L = 0.$$

$$\text{Case F: } Y_S = \infty, \quad Y_L = \infty.$$

The notation is shown in Figure 11.

Case A

For this case, $Y_S = \infty$ and $Y_L = 0$. Setting the admittance across the output port to zero, the relationship

$$Y_{22} = 0$$

is obtained. Therefore, the network is unstable if y_{22} has a positive real zero. However, y_{22} may have at most one positive real pole, if the first critical frequency of y_{22} from the right is a pole. For the remainder of this chapter, all critical frequencies will be counted from the right.

To find the natural frequencies in the input loop, consider the input loop impedance:

$$z_{11} = \frac{y_{22}}{y_{11}y_{22} - y_{12}^2} = 0.$$

If the admittance functions are expressed as

$$y_{11} = \frac{P_1(s)}{Q(s)}$$

$$y_{12} = \frac{P_0(s)}{Q(s)}$$

$$y_{22} = \frac{P_2(s)}{Q(s)}$$

then

$$y_{11}y_{22} - y_{12}^2 = \frac{P_1(s)P_2(s) - [P_0(s)]^2}{[Q(s)]^2}$$

If $k_{11}k_{22} = k_{12}^2$ with $k_{12} \neq 0$ in each pole, i.e., if the network is compact in each pole, then $Q(s)$ is a factor of the numerator of $y_{11}y_{22} - y_{12}^2$, and

$$y_{11}y_{22} - y_{12}^2 = \frac{P_3(s)}{Q(s)}. \quad (13)$$

Then

$$z_{11} = \frac{P_2(s)}{P_3(s)}$$

It is seen that setting z_{11} to zero requires that y_{22} have no zeros on the positive real axis, and the natural frequencies in the input loop

are the same as those that appear at the output. The only constraint on y_{11} and y_{12} is that the network be compact in each pole. Since y_{22} may have at most one positive real pole, y_{11} and y_{12} may also have at most one positive real pole. There is no constraint on the zeros of y_{11} , other than that y_{11} must be $\pm R, C$; therefore y_{11} may have at most two positive real zeros.

If the admittance functions are not compact in all of the poles, these poles do not appear as a factor of the numerator of $y_{11}y_{22} - y_{12}^2$. Thus,

$$y_{11}y_{22} - y_{12}^2 = \frac{P_4(s)}{Q(s) Q_1(s)} \quad (14)$$

where

$$Q_1(s) = \prod (s + s_j),$$

and s_j are the poles in which the admittance functions are not compact. Then

$$z_{11} = \frac{P_2(s) Q_1(s)}{P_4(s)}.$$

Therefore, y_{11} , y_{12} , and y_{22} may have no noncompact poles on the positive real axis. This is an example of a natural frequency appearing in the input loop that does not appear at the output.

Case B

For this case, $Y_S = \infty$ and $Y_L = G_L$. By setting the admittance across the output port to zero, the relationship

$$y_{22} + G_L = 0$$

is obtained. The natural frequencies are therefore the zeros of $(y_{22} + G_L)$. Consider a sketch of y_{22} versus σ , shown in Figure 12. The origin is purposely omitted from the sketch. The addition of G_L to y_{22} shifts the positions of the zeros to the left on the σ axis,

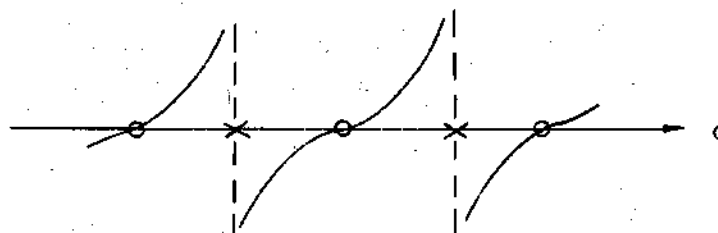


Figure 12. Sketch of y_{22} versus σ

but a zero cannot be shifted beyond the adjacent pole location. If the first critical frequency of y_{22} is zero, this zero may occur on the positive real axis, but the first pole must occur on the negative real axis. If the first critical frequency of y_{22} is a pole, the second pole of y_{22} must occur on the negative real axis. Therefore, y_{22} may have at most one zero and one pole on the positive real axis.

To find the natural frequencies in the input loop, consider the input impedance. For this case,

$$z_{11} = \frac{1}{Y_i} = \frac{1}{y_{11} - \frac{y_{12}^2}{y_{22} + G_L}} = \frac{y_{22} + G_L}{y_{11}G_L + y_{11}y_{22} - y_{12}^2} = 0.$$

From the discussion of Case A, if the network is compact in each pole, then the zeros of z_{11} are the zeros of $(y_{22} + G)$. Thus, the only

constraint on y_{11} and y_{12} is that the network be compact in each pole. And since y_{22} may have at most one positive real pole, y_{11} and y_{12} may have at most one positive real pole. There is no constraint on the zeros of y_{11} , other than that y_{11} must be $\pm R, C$; therefore, y_{11} may have at most two positive real zeros.

If the admittance functions are not compact in all of the poles, $(y_{11}y_{22} - y_{12}^2)$ is given by (14). The factor $Q_1(s)$ appears in the numerator of z_{11} . Therefore, y_{11} , y_{12} , and y_{22} may have no non-compact poles on the positive real axis.

Case C

For this case, $Y_S = G_S$ and $Y_L = G_L$. Consider the network of Figure 11. For this network,

$$Y_i = y_{11} - \frac{y_{12}^2}{y_{22} + Y_L}$$

The natural frequencies of the input circuit are the zeros of

$$Y_S + Y_i = G_S + \frac{y_{11}G_L + y_{11}y_{22} - y_{12}^2}{y_{22} + G_L} \quad (15)$$

If y_{11} , y_{12} , and y_{22} are compact in each pole, then, from (13), the polynomial $Q(s)$ does not appear as a factor in either the numerator or the denominator of Y_i . Therefore, the zeros of Y_i are the zeros of $(y_{11}G_L + y_{11}y_{22} - y_{12}^2)$, and the poles of Y_i are the zeros of $(y_{22} + G_L)$. The zeros of (15) are always to the left of the zeros of Y_i on the real axis, since Y_i is a $\pm R, C$ admittance. The first zero of Y_i may occur

on the positive real axis, but the first pole to the left of this zero must occur on the negative real axis.

If the first critical frequency of Y_1 is a zero, the first zero of $(y_{22} + G_L)$, which is the first pole of Y_1 , must occur on the negative real axis and hence the first zero of y_{22} may occur on the positive real axis. If the first critical frequency of y_{22} is a pole, y_{22} may have one zero and one pole on the positive real axis.

If the first critical frequency of Y_1 is a pole, the second zero of $(y_{22} + G_L)$, which is the second pole of Y_1 , must occur on the negative real axis. If the first critical frequency of y_{22} is a pole, y_{22} may have two zeros and two poles on the positive real axis.

If y_{11} , y_{12} , and y_{22} are not compact in each pole, then, from (14) and (15), the factor $Q_1(s)$ appears in the denominator of Y_1 . Therefore, the poles of Y_1 are now the zeros of $(y_{22} + G_L)$ plus the poles of y_{11} , y_{12} , and y_{22} for which these functions are not compact. If the first critical frequency of Y_1 is a zero, the admittance functions may have no poles on the positive real axis which are not compact. However, the same limitations are present on y_{22} as in the compact case. If the first critical frequency of Y_1 is a pole, this pole may be a pole of y_{11} , y_{12} , and y_{22} which is not compact. However, there may be only one such pole on the positive real axis. If a noncompact pole is present on the positive real axis, y_{22} may have at most one zero on the positive real axis.

The same statements are true regarding y_{11} , since Y_S and Y_L are both conductances. A numerical example illustrating Case C is given in Appendix III.

Case D

For this case, $Y_S = 0$ and $Y_L = G_L$. Setting the impedance of the output loop equal to zero, the relationship

$$z_{22} + \frac{1}{G_L} = 0$$

is obtained. Now

$$z_{22} = \frac{Y_{11}}{Y_{11}Y_{22} - Y_{12}^2}.$$

From (13), if y_{11} , y_{12} , and y_{22} are compact in each pole, the zeros of y_{11} are the zeros of z_{22} . Since z_{22} is a $\pm R, C$ impedance, it has the shape shown in Figure 13. The addition of $1/G_L$ to z_{22} will shift the positions of the zeros to the right. For the network to be stable, the first zero of z_{22} must occur on the negative real axis. Therefore, the first zero of y_{11} must occur on the negative real axis. If the first critical frequency of y_{11} is a pole, this pole may occur on the positive real axis.

If y_{11} , y_{12} , and y_{22} are not compact in each pole, $Y_{11}Y_{22} - Y_{12}^2$ is given by (14), and $Q_1(s)$ is a factor of the numerator of z_{22} . Therefore, the poles for which y_{11} , y_{12} , and y_{22} are not compact are the zeros of z_{22} , and these poles must occur on the negative real axis.

For the restrictions on y_{22} , consider Y_i in Equation (15). Since Y_S of (15) is zero, the zeros of Y_i are the natural frequencies of the network. If the admittance functions are compact in each

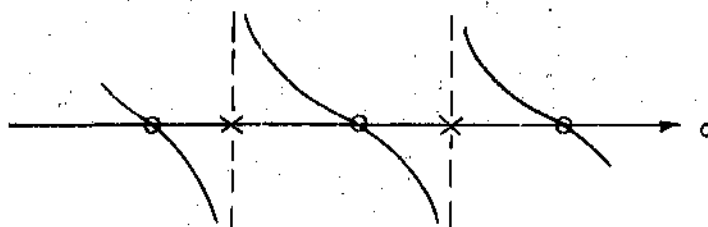


Figure 13. Sketch of z_{22} versus σ

pole, the poles of Y_i are the zeros of $(y_{22} + G_L)$. If the first critical frequency of Y_i is a zero, the first pole of Y_i must occur on the negative real axis. Then y_{22} may have one zero on the positive real axis. If, in addition, the first critical frequency of y_{22} is a pole, y_{22} may also have one pole on the positive real axis. If the first critical frequency of Y_i is a pole, then the second zero of $(y_{22} + G_L)$ must occur on the negative real axis. Then y_{22} may have two zeros on the positive real axis. However, since y_{11} is limited to one pole on the positive real axis, y_{22} may have only one pole there. A numerical example illustrating Case D is given in Appendix III.

Case E

For this case, $Y_S = 0$ and $Y_L = 0$. From (15), setting the input admittance equal to zero, the relationship

$$Y_i = \frac{y_{11}y_{22} - y_{12}^2}{y_{22}} = 0$$

is obtained. For the network compact, the zeros of Y_1 are the zeros of $Y_{11}Y_{22} - Y_{12}^2$, and the poles are the zeros of y_{22} . If the first critical frequency of Y_1 is a zero, the first zero of y_{22} must occur on the negative real axis. Therefore, y_{22} may have at most one pole on the positive real axis. If the first critical frequency of Y_1 is a pole, the second zero of y_{22} must occur on the negative real axis. Then y_{22} may have at most one zero and two poles on the positive real axis. Since the terminations of the network are both open circuits, the same statements apply to y_{11} .

If the network is not compact in all poles, the poles in which the network is not compact are also the poles of Y_1 , as is seen from (14). If the first critical frequency of Y_1 is a zero, no noncompact poles may occur on the positive real axis. If the first critical frequency of Y_1 is a pole, this pole may occur on the positive real axis, and this pole may be a noncompact pole. There may be no zero on the positive real axis for this case. A numerical example illustrating Case E is given in Appendix III.

Case F

For this case, $Y_S = \infty$ and $Y_L = \infty$. The natural frequencies of the input loop are the roots of

$$\frac{1}{y_{11}} = 0.$$

Thus, y_{11} can have no poles on the positive real axis, but may have one zero there. Since Y_S and Y_L are the same, the same statement applies to y_{22} .

Conclusions

The functions y_{11} and y_{22} may have at most two zeros and two poles on the positive real axis with the resulting network stable. This condition can occur in a compact network when both the source and the load admittances are finite conductances.

CHAPTER V

SYNTHESIS OF VOLTAGE TRANSFER FUNCTIONS FOR SINGLY-LOADED NETWORKS

The synthesis procedures developed in Chapter II will now be applied to the synthesis of three different transfer functions. The first of these functions, the voltage transfer function for the singly-loaded network, will be considered in this chapter. The other two transfer functions, the voltage transfer functions for the open-circuited network and for the double-loaded network, will be considered in the next two chapters.

Necessary Conditions for Y_{12}

For the network of Figure 14,

$$Y_{12} = \frac{I_2}{E_1} = -\frac{E_2}{E_1} = \frac{y_{12}}{1 + y_{22}}$$

The necessary conditions of Y_{12} can be obtained by investigating the necessary conditions on y_{12} and y_{22} . For y_{12} and y_{22} to be realizable as $\pm R, C$ functions, these functions can be expressed as

$$\begin{aligned} y_{12} &= \frac{p(s)}{q_2(s)} \\ y_{22} &= \frac{q_1(s)}{q_2(s)} \end{aligned} \tag{16}$$

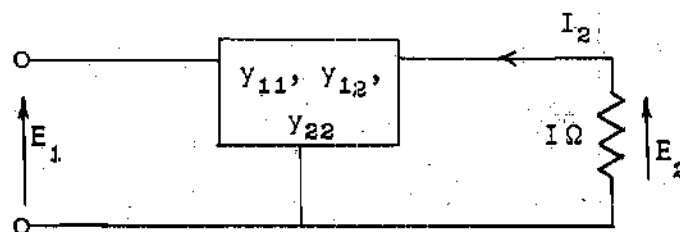


Figure 14. Network for Synthesizing Y_{12}

where $p(s)$, $q_1(s)$, and $q_2(s)$ are polynomials in s . The coefficients of $p(s)$ must be real, and the zeros of $q_1(s)$ and $q_2(s)$ must alternate on the real axis. Now,

$$Y_{12} = \frac{\frac{p(s)}{q_2(s)}}{1 + \frac{q_1(s)}{q_2(s)}} = \frac{p(s)}{q_1(s) + q_2(s)} = \frac{p(s)}{q(s)} \quad (17)$$

Since $q(s)$ is the sum of $q_1(s)$ and $q_2(s)$, it is seen from Figure 15 that $q(s)$ can have zeros only along the real axis. And in order that the resulting network be stable, $q(s)$ can have zeros only on the negative real axis.

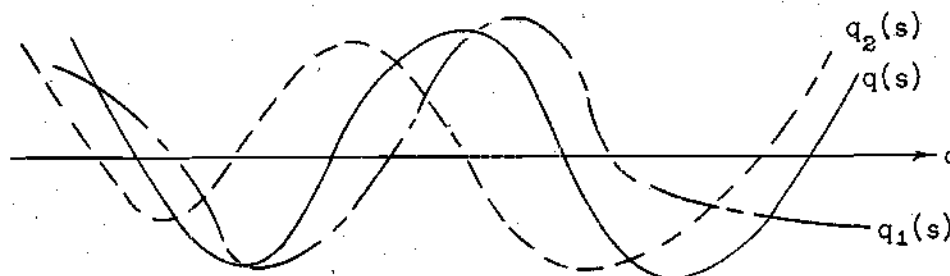


Figure 15. Plot of $q(s)$, $q_1(s)$, and $q_2(s)$

It is also seen from Figure 15 that $q(s)$ may be of one degree less than $q_1(s)$ and $q_2(s)$ but may not be lesser in degree than this. If $p(s)$ is of the same degree as $q_2(s)$, $p(s)$ is of one degree greater than $q(s)$. The degree of $p(s)$ may not be more than one greater than that of $q(s)$, since this condition requires y_{12} to have poles that are not present in y_{22} .

In summary, the necessary conditions for the synthesis of $Y_{12} = p(s)/q(s)$ by a stable $\pm R, C$ network are:

1. The coefficients of $p(s)$ are real.
2. The zeros of $q(s)$ occur only on the negative real axis.
3. The degree of $p(s)$ is not more than one greater than that of $q(s)$.

It will be shown by the synthesis procedure of the next section that these conditions are also sufficient.

Method of Synthesis

Given Y_{12} , the zeros of $q_2(s)$ must be chosen such that y_{22} is a $\pm R, C$ admittance. Let

$$q_2(s) = k \prod (s + s_i) .$$

The polynomial $q_2(s)$ can be chosen to have one zero less than $q(s)$, the same number of zeros as $q(s)$, or one zero more than $q(s)$. If the degree of $p(s)$ is one greater than the degree of $q(s)$, $q_2(s)$ must have one zero more than $q(s)$. Otherwise, any of the three choices may be made.

If it is desired that $q_2(s)$ have one zero less than $q(s)$, the right-most zero of $q_2(s)$ should be chosen to the left of the right-most zero of $q(s)$, and the other zeros should be chosen to alternate with those of $q(s)$. No zero should be chosen to the left of the left-most zero of $q(s)$. The constant k should be made positive. Under these conditions, y_{22} will be a $\pm R, C$ admittance. This can be seen by first considering $q_1(s) = [q(s) - q_2(s)]$. From Figure 15, it is seen that the zeros of $q_1(s)$ and $q(s)$ alternate. Furthermore, in the vicinity of the right-most zero of $q_2(s)$, $q_1(s)$ is negative. The variation of $q_1(s)/q_2(s)$ in the vicinity of this zero is as shown in Figure 16. This is the manner in which a $\pm R, C$ admittance must vary, and $q_1(s)/q_2(s)$ is a $\pm R, C$ admittance.

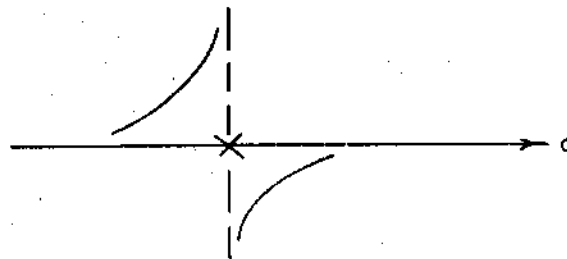


Figure 16. Variation of $q_1(s)/q_2(s)$ in the Vicinity of a Zero of $q_2(s)$

If it is desired that $q_2(s)$ have the same number of zeros as $q(s)$, the right-most zero of $q_2(s)$ can be chosen to the left of the right-most zero of $q(s)$, and the other zeros chosen to alternate with those of $q(s)$. The constant k should be made positive. Then, by

the above discussion, it is seen that $q_1(s)/q_2(s)$ is a $\pm R, C$ admittance. Or, the right-most zero of $q_2(s)$ can be chosen to the right of the right-most zero of $q(s)$. For this choice, k should be made negative. Then it is seen that $q_1(s)/q_2(s)$ is also a $\pm R, C$ admittance.

If it is desired that $q_2(s)$ have one more zero than $q(s)$, the right-most zero of $q_2(s)$ should be chosen to the right of the right-most zero of $q(s)$, and k should be made negative. The remaining zeros of $q_2(s)$ should be chosen to alternate with those of $q(s)$. Then $q_1(s)$, which is equal to the difference of $q(s)$ and $q_2(s)$, will have the variation shown in Figure 17. An examination of $q_1(s)/q_2(s)$ in the vicinity of any of the zeros of $q_2(s)$ shows that this function is a $\pm R, C$ admittance.

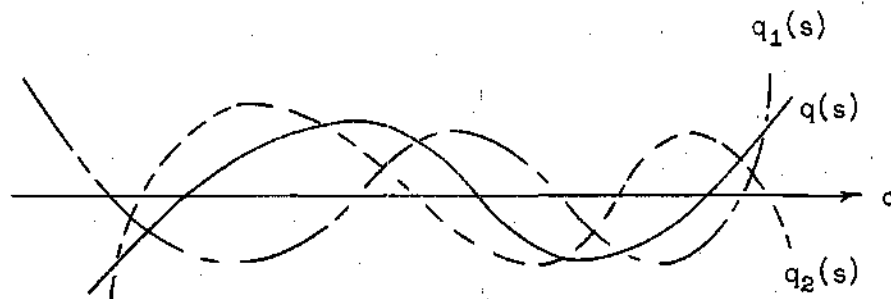


Figure 17. Variation of $q(s)$, $q_1(s)$, and $q_2(s)$

With the zeros of $q_2(s)$ chosen, the functions y_{12} and y_{22} will be of the form

$$\begin{aligned}
 y_{12} &= g_{12} + k_{12\infty}s + \frac{k_{120}}{s} + \sum \frac{k_{12i}s}{s+s_i} \\
 y_{22} &= g_{22} + k_{22\infty}s + \frac{k_{220}}{s} + \sum \frac{k_{22i}s}{s+s_i}
 \end{aligned} \tag{18}$$

The zeros of $q_2(s)$ can be chosen such that $k_{12\infty}$ and k_{120} are always zero. It is desirable that both $k_{12\infty}$ and k_{120} be zero, because these terms in general will require additional negative resistors. To make $k_{12\infty}$ zero, the degree of $q_2(s)$ should be at least as great as that of $p(s)$. To make k_{120} zero, $q_2(s)$ should not have a zero at the origin.

Once y_{11} and y_{22} are determined, the circuit can be synthesized by the straightforward procedure of synthesizing each pole separately given in Chapter II. The conductance terms can be realized with two resistors as shown in Figure 18. Both of these resistors may be negative for a specific Y_{12} .

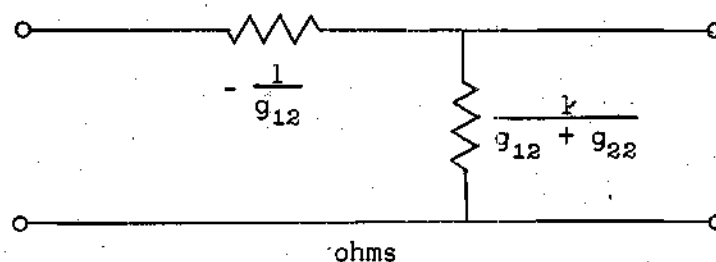


Figure 18. Network for Synthesizing Conductance Terms

The different cases possible for the finite poles will now be considered. Suppose that s_i is positive. If k_{12} for this pole is negative and less than or at most equal to k_{22} in magnitude, a value of k_{11} can be chosen for y_{11i} such that the pole is compact, and the network for this pole will require a negative resistor across the input terminals. Since y_{11} does not affect Y_{12} , this negative

resistor can be omitted. If k_{12} is negative and greater in magnitude than k_{22} , a value of k_{11} can be chosen for y_{11i} such that the pole is compact. Then the network for this pole will require a negative resistor across the output terminals. If k_{12} is positive, a value of k_{11} can be chosen for y_{11i} such that the pole is compact. Then the network for this pole will require a negative resistance across the top terminals of the network. Thus, for the synthesis of the negative real axis poles and the conductance terms, a maximum of two negative resistors will be required.

In order that the network be stable, $q(s)$ can have zeros only on the negative real axis. The zeros of $q_2(s)$ can always be chosen on the negative real axis. This choice is desirable, since each pole on the positive real axis will require one internal negative resistor. For this reason, the synthesis of poles on the positive real axis has not been considered.

In conclusion, a maximum of two negative resistors are required for the synthesis of a stable, $\pm R, C$ voltage transfer function for a singly-loaded network.

Methods of Reducing the Number of Required Negative Resistors

The number of negative resistors required to synthesize a specific Y_{12} sometimes be reduced by the proper choice of zero locations and the constant multiplier of $q_2(s)$.

Let Y_{12} be expressed as

$$Y_{12} = \frac{a_0 + a_1s + \dots + a_ms^m}{b_0 + b_1s + \dots + s^n} \quad (19)$$

Inspection of (16), (17), (18), and (19) shows that

$$g_{12} = \frac{a_0}{k \prod s_i}$$

$$g_{22} = \frac{b_0 - k \prod s_i}{k \prod s_i}$$

where s_i are the zero locations and k is the constant multiplier of $q_2(s)$. If g_{12} is positive, a negative resistor will be required across the top terminals of the network. If, in addition, g_{22} is negative and greater in magnitude than g_{12} , a negative resistor is also required across the output terminals of the network. The sign of $\prod s_i$ is always positive, but k may be positive or negative. If $q_2(s)$ is chosen to have one less zero than $q(s)$, k is positive. If $q_2(s)$ is chosen to have the same number of zeros as $q(s)$, k may be positive or negative. If $q_2(s)$ is chosen to have one more zero than $q(s)$, k is negative. If the degree of $p(s)$ is one greater than that of $q(s)$, k must be negative. However, if the degree of $p(s)$ is equal to or less than that of $q(s)$, k may be chosen positive or negative. Thus, there is some possibility of varying the sign of k to reduce the number of negative resistors required to realize the conductance terms.

If two negative resistors are required for the realization of the conductance terms, any further effort to reduce the number of negative resistors required is futile. However, if both negative resistors are

not required, then possibly a reduction can be effected. Consider the poles of y_{12} and y_{22} of (19). If k_{12} is negative and less than or equal to k_{22} in magnitude for any of these poles, no negative resistor is required in the synthesis of this pole. If this condition can be satisfied for every pole for which k_{12} is negative, the synthesis of the poles will not require a negative resistor across the output.

The possibility of this occurrence can be determined by the following procedure. Now

$$k_{12i} = \frac{p(s_i)}{s_i q_2'(s_i)}$$

where

$$q_2'(s) = \frac{dq_2(s)}{ds}.$$

Also,

$$k_{22i} = \frac{q_1(s_i)}{s_i q_2'(s_i)} = \frac{q(s_i)}{s_i q_2'(s_i)}$$

since

$$q(s_i) = q_1(s_i) + q_2(s_i) = q_1(s_i).$$

Thus,

$$\frac{k_{12i}}{k_{22i}} = \frac{p(s_i)}{q(s_i)}.$$

It is desirable that this ratio be negative and less than or equal to unity in magnitude. If $p(s)/q(s)$ is calculated as a function of $s = \sigma$, the intervals over which this condition is satisfied become evident, and the zero locations can be chosen in these intervals

whenever possible. The zero locations should be chosen where $p(s)/q(s)$ is equal to (-1) , if possible, in order to save circuit elements.

If $m < n$ in (19), and if the synthesis of the conductance terms will allow k to be positive, then $q_2(s)$ should be chosen to have one zero less than $q(s)$. This choice will give y_{22} a pole at infinity, but y_{12} will not have a pole there. One pole is saved in y_{12} , and fewer circuit elements are required in the synthesizing network. If $m = n$, if the synthesis of the conductance terms will allow k to be positive, and if the synthesis of the resulting pole of y_{12} at infinity does not require a negative resistor, then $q_2(s)$ should be chosen to have one zero less than $q(s)$. At $s = \infty$,

$$\frac{k_{12}}{k_{22}} = a_m$$

where a_m is given in (19). If a_m is negative and less than or equal to unity in magnitude, no negative resistor is required in the synthesis of this pole. A numerical example illustrating the synthesis of y_{12} is given in Appendix III.

CHAPTER VI

SYNTHESIS OF OPEN-CIRCUIT VOLTAGE TRANSFER FUNCTIONS

The synthesis procedures developed in Chapter II will now be applied to the synthesis of the voltage transfer function for the open circuited network.

Necessary Conditions for the Open-Circuit Voltage Transfer Function

For the network of Figure 19, the voltage transfer function is

$$T(s) = \frac{E_2(s)}{E_1(s)} = - \frac{y_{12}(s)}{y_{22}(s)} \quad (20)$$

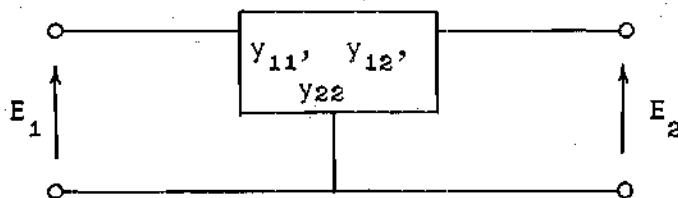


Figure 19. Network for Synthesizing the Voltage Transfer Function

The necessary conditions for $T(s)$ can be determined from the investigation of the necessary conditions for y_{12} and y_{22} . These functions can be expressed as

$$\begin{aligned} y_{12} &= \frac{P(s)}{R(s)} \\ y_{22} &= \frac{Q(s)}{R(s)} \end{aligned} \quad (21)$$

where $P(s)$, $Q(s)$, and $R(s)$ are polynomials in s . For y_{12} and y_{22} to be realizable as $\pm R, C$ twoports, the coefficients of $P(s)$ must be real and the zeros of $Q(s)$ and $R(s)$ must alternate on the real axis. Now, from (20) and (21),

$$T(s) = \frac{E_2(s)}{E_1(s)} = - \frac{P(s)}{Q(s)} = \frac{a_0 + a_1s + \dots + a_ms^m}{b_0 + b_1s + \dots + s^n} \quad (22)$$

For the resulting network to be stable, the zeros of $Q(s)$ must fall on the negative real axis. Since y_{12} cannot have poles that do not appear in y_{22} , and since $R(s)$ can be at most one degree higher than $Q(s)$, $P(s)$ can be at most one degree higher than $Q(s)$.

In summary, the necessary conditions on $T(s)$ for realizability as a stable $\pm R, C$ network are:

1. The coefficients of $P(s)$ are real.
2. The zeros of $Q(s)$ may occur only on the negative real axis.
3. The degree of $P(s)$ is not more than one greater than that of $Q(s)$.

It will be shown in the next section that these conditions are also sufficient.

Synthesis Procedure

Let $R(s)$ be expressed as

$$R(s) = k \prod (s + s_i).$$

The zeros of $R(s)$ must be chosen to alternate with those of $Q(s)$.

The method of choosing the zeros of $R(s)$ is the same as that used in

the synthesis of Y_{12} in Chapter V. The functions y_{12} and y_{22} can be expressed as

$$y_{12} = \frac{P(s)}{R(s)} = g_{12} + k_{12\infty}s + \frac{k_{120}}{s} + \sum \frac{k_{12i}}{s + s_i}$$

$$y_{22} = \frac{Q(s)}{R(s)} = g_{22} + k_{22\infty}s + \frac{k_{220}}{s} + \sum \frac{k_{22i}}{s + s_i}$$

As was shown for the development for Y_{12} in Chapter V, $k_{12\infty}$, $k_{22\infty}$, k_{120} , and k_{220} can always be made zero. It is desirable that these terms be zero, since, in general, the realization of these terms requires additional negative resistors. The conductance terms can be realized by the network of Figure 18. Now,

$$g_{12} = \frac{-a_0}{k \prod s_i}$$

$$g_{22} = \frac{b_0}{k \prod s_i}$$

where a_0 and b_0 are defined in (22). The ratio of g_{12} and g_{22} is independent of the zero locations of $R(s)$. However the signs of g_{12} and g_{22} can be changed by changing the sign of k . As was the case for Y_{12} , if $R(s)$ is chosen to have one zero less than $Q(s)$, then k must be positive for y_{22} to be a $\pm R, C$ admittance. If $R(s)$ is chosen to have the same number of zeros as $Q(s)$, k may be positive or negative. If the first zero of $R(s)$ is to the left of the right-most zero of $Q(s)$, k is positive. If the first zero of $R(s)$ is to the right of the first zero of $Q(s)$, k is negative. If $R(s)$

is chosen to have one zero more than $Q(s)$, k must be negative. Thus, the possibility exists for reducing the number of negative resistors required to realize the conductance terms by the proper choice of the sign of k .

The poles on the finite real axis can be synthesized by the methods used for Y_{12} . As was shown for Y_{12} , the possibility of reducing the number of required negative resistors by the proper choice of the zero locations of $R(s)$ exists. Now,

$$k_{12i} = \frac{P(s_i)}{s_i R'(s_i)}$$

$$k_{22i} = \frac{Q(s_i)}{s_i R'(s_i)}$$

Thus,

$$\frac{k_{12i}}{k_{22i}} = \frac{P(s_i)}{Q(s_i)}$$

The methods of choosing the zeros of $R(s)$ such as to reduce the number of negative resistors required are then seen to be the same as those of Y_{12} . If $m < n$, and if the synthesis of the conductance terms will allow k to be positive, $R(s)$ should be chosen to have one less zero than $Q(s)$. This choice will give y_{22} a pole at infinity, but y_{12} will not have a pole there. One pole is saved in y_{12} , and fewer circuit elements are required in the synthesizing network. If $m = n$, if the synthesis of the conductance terms will allow k to be positive,

and if the realization of the resulting pole at infinity does not require negative resistors, then $R(s)$ should be chosen to have one zero less than $Q(s)$. At $s = \infty$,

$$\frac{k_{12}}{k_{22}} = a_m$$

where a_m is given in (22). If a_m is negative and less than or equal to one in magnitude, no negative resistor is required for this pole.

Thus, at most, two negative resistors are required to synthesize a given $\pm R, C$ voltage transfer function for the open-circuited network.

A numerical example illustrating the synthesis of the open-circuit voltage transfer function is given in Appendix III.

CHAPTER VII

SYNTHESIS OF VOLTAGE TRANSFER FUNCTIONS FOR DOUBLE-LOADED NETWORKS

A method of synthesis of the voltage transfer function E_2/E_1 for the network shown in Figure 20, where R_1 and R_2 are specified resistance values, is to be devised using the synthesis procedures of Chapter II.

Necessary Conditions for the Double- Loaded Voltage Transfer Function

Consider the network of Figure 20, which is redrawn in Figure 21.

For this network,

$$T(s) = \frac{E_2(s)}{E_1(s)} = -\frac{I_2(s)R_2}{E_1(s)} = -y_{12}(s)R_2.$$

Or,

$$y_{12}(s) = -\frac{1}{R_2} T(s) = \frac{a_0 + a_1s + \dots + a_ms^m}{b_0 + b_1s + \dots + s^n} \quad (23)$$

The necessary conditions for $T(s)$ can be obtained from the necessary conditions for y_{12} . For y_{12} to be realizable as a $\pm R, C$ admittance, and for the resulting network to be stable, the following conditions must be satisfied:

1. $T(s)$ can have poles only on the negative real axis.
2. The coefficients a_0, a_1, \dots, a_m must be real.

3. The degree of the numerator of $I(s)$ is at most one greater than the degree of the denominator.

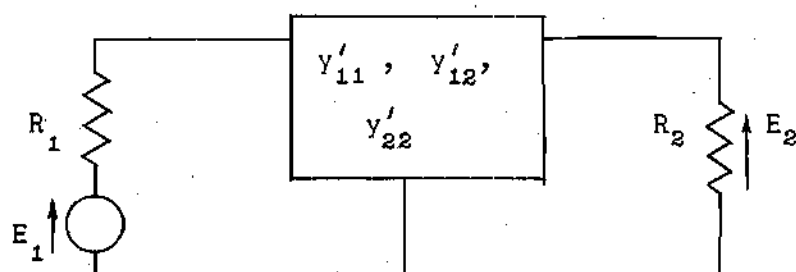


Figure 20. The Double-Loaded Network

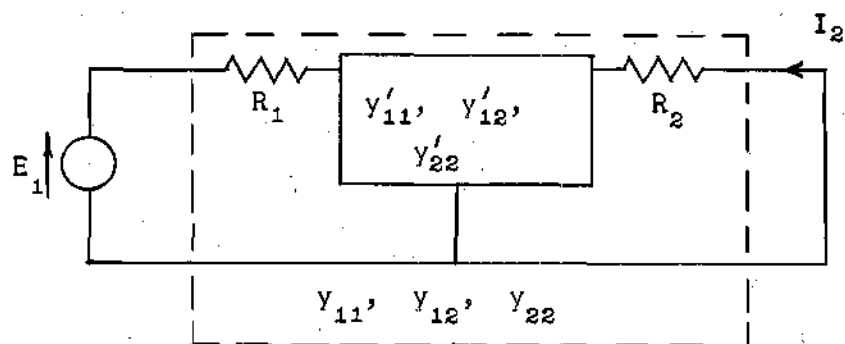


Figure 21. Notation used for the Double-Loaded Network.

These necessary conditions will be shown to be sufficient also by the synthesis procedures of the following sections.

Basic Synthesis Procedures

The poles of y_{12} must also be the poles of y_{11} and y_{22} .

Thus, y_{11} and y_{22} can be written as

$$\begin{aligned}
 y_{11}(s) &= \frac{c_0 + c_1s + \dots + c_p s^p}{b_0 + b_1s + \dots + s^n} \\
 y_{22}(s) &= \frac{d_0 + d_1s + \dots + d_p s^p}{b_0 + b_1s + \dots + s^n}
 \end{aligned}
 \tag{24}$$

where

$$n - 1 \leq p \leq n + 1.$$

There are several different methods by which $T(s)$ can be synthesized. A direct method is to assume the residues in the poles of y_{11}/s and y_{22}/s equal in magnitude to the residues in the respective poles of y_{12}/s . Then this set of admittances can be realized by the synthesizing networks of Chapter II. If $m < (n + 1)$, only one negative resistor is required in this realization. This negative resistor appears across the top terminals of the network. If $m = (n + 1)$, then possibly two additional negative resistors are required to realize the pole at infinity. To realize the resistances R_1 and R_2 , resistors of values R_1 and $(-R_1)$ are connected in series in the input circuit, and resistors of values R_2 and $(-R_2)$ are connected in series in the output circuit. This synthesis procedure requires three negative resistors if y_{12} has no pole at infinity, and either three or five negative resistors if this pole is present.

A second method, which is similar to the synthesis method of Kuh (12) for RC networks, is to choose y_{11} and y_{22} such that R_1 and R_2 are extractable from z_{11} and z_{22} , respectively. The functions z_{11} and z_{22} are the equivalent open-circuit impedances of the admittance functions y_{11} , y_{12} , and y_{22} . This method is based on

the following derivations. If the short-circuit admittances are converted to the equivalent open-circuit impedances, then, considering Figure 21,

$$\begin{aligned} z_{11} &= \frac{y_{22}}{|y|} = z'_{11} + R_1 \\ z_{12} &= -\frac{y_{12}}{|y|} = z'_{12} \\ z_{22} &= \frac{y_{11}}{|y|} = z'_{22} + R_2 \end{aligned} \quad (25)$$

where

$$|y| = y_{11}y_{22} - y_{12}^2.$$

Then

$$\begin{aligned} y'_{11} &= \frac{z'_{22}}{|z'|} \\ y'_{12} &= -\frac{z'_{12}}{|z'|} \\ y'_{22} &= \frac{z'_{11}}{|z'|} \end{aligned}$$

where

$$|z'| = z'_{11}z'_{22} - z'_{12}^2.$$

The functions y'_{11} , y'_{12} , and y'_{22} can be realized by the synthesizing networks of Chapter II.

It will be shown that, except for the case that y_{12} has a pole at infinity, this second method will require at most three negative resistors, and in some cases only one or two. If y_{12} has a pole at

infinity, this method will require at most five negative resistors, and in some cases as few as one. This method is developed in the following sections, and the conditions for which this method requires fewer negative resistors than the first method are presented. In terms of the degree of the numerator of $T(s)$, this method is divided into three cases.

These cases are:

1. $m < n$.
2. $m = n$.
3. $m = n + 1$.

Case 1

Consider the admittance functions y_{11} , y_{12} , and y_{22} expressed in the following manner,

$$\begin{aligned}
 y_{11} &= \sum \frac{k_{11i}s}{s + s_i} + g_{11} = \frac{c_0 + c_1s + \dots + c_ns^n}{b_0 + b_1s + \dots + s^n} \\
 y_{12} &= \sum \frac{k_{12i}s}{s + s_i} + g_{12} = \frac{a_0 + a_1s + \dots + a_ms^m}{b_0 + b_1s + \dots + s^n} \\
 y_{22} &= \sum \frac{k_{22i}s}{s + s_i} + g_{22} = \frac{d_0 + d_1s + \dots + d_ns^n}{b_0 + b_1s + \dots + s^n}
 \end{aligned} \tag{26}$$

in which y_{12} , from (23), is known. The residues in the poles of y_{11}/s and y_{22}/s are chosen such that these poles are compact. Then

$$|y| = \frac{b'_0 + b'_1s + \dots + b'_ns^n}{b_0 + b_1s + \dots + s^n} \tag{27}$$

where

$$b'_n = c_n d_n \quad (28)$$

When z_{11} is expressed as

$$z_{11} = \frac{c_0 + c_1 s + \dots + c_n s^n}{b'_0 + b'_1 s + \dots + b'_n s^n}$$

the ratio c_n/b'_n will be defined as the extractable resistance of z_{11} . A similar definition will apply to z_{22} as well. If the extractable resistances are removed from z_{11} and z_{22} in (25), then, as $s \rightarrow \infty$, z'_{11} and z'_{22} go to zero. Therefore,

$$z_{11} \Big|_{s=\infty} = \frac{y_{22}}{|y|} \Big|_{s=\infty} = \frac{d_n}{b'_n} = \frac{1}{c_n} = R_1 \quad (29)$$

$$z_{22} \Big|_{s=\infty} = \frac{y_{11}}{|y|} \Big|_{s=\infty} = \frac{c_n}{b'_n} = \frac{1}{d_n} = R_2$$

From (26) and (29)

$$c_n = \sum k_{11i} + g_{11} = \frac{1}{R_1} \quad (30)$$

$$d_n = \sum k_{22i} + g_{22} = \frac{1}{R_2}$$

If g_{11} and g_{22} and the residues in the poles of y_{11}/s and y_{22}/s are chosen such that (30) is satisfied, R_1 and R_2 are extractable from z_{11} and z_{22} , respectively.

In order that a method can be devised by which the number of negative resistors used is less than three, consider y'_{11} , y'_{12} ,

and y'_{22} . In general,

$$\begin{aligned} y'_{11} &= \sum \frac{k'_{11s}}{s + s_i} + g'_{11} + k'_{11\infty}s + \frac{k'_{110}}{s} \\ y'_{12} &= \sum \frac{k'_{12s}}{s + s_i} + g'_{12} + k'_{12\infty}s + \frac{k'_{120}}{s} \\ y'_{22} &= \sum \frac{k'_{22s}}{s + s_i} + g'_{22} + k'_{22\infty}s + \frac{k'_{220}}{s} \end{aligned} \quad (31)$$

Only one negative resistor is required to realize this set of admittances if the following conditions are satisfied:

- a. $k'_{11i} = k'_{22i} = |k'_{12i}|$.
- b. $g'_{11} \geq -g'_{12}$; $g'_{22} \geq -g'_{12}$.
- c. $s_i > 0$.
- d. $k'_{12\infty} = 0$.
- d. k'_{110} , k'_{120} , and k'_{220} are all zero.

This is because the negative resistor across the top terminals of the network can be used, if necessary, for both the conductance terms and for the poles for which k'_{12i} is positive. No negative resistors are required for the poles for which k'_{12i} is negative.

Consider Condition a first. If, in (26), k_{11i} and k_{22i} are made equal in magnitude to k_{12i} , if g_{11} and g_{22} are chosen equal, and if equal values of resistance are extracted from z_{11} and z_{22} , then y'_{11} and y'_{22} are equal and Condition a is satisfied.

The value of g_{11} is then chosen to give a maximum value of extractable resistance without violating Condition b. To find this

value of g_{11} , consider y_{11} , y_{12} , and y_{22} at $s = 0$. Let R be the maximum value of extractable resistance. Then, from (26) and (30),

$$y_{11}|_{s=0} = g_{11} = \frac{1}{R} - \sum |k_{12i}|$$

$$y_{12}|_{s=0} = g_{12} \quad (32)$$

$$y_{22}|_{s=0} = g_{22} = \frac{1}{R} - \sum |k_{12i}| = g_{11}.$$

Then

$$z_{11}|_{s=0} = z_{22}|_{s=0} = z'_{11}|_{s=0} + R = \frac{g_{11}}{|y|} \quad (33)$$

$$z_{12}|_{s=0} = z'_{12}|_{s=0} = -\frac{g_{12}}{|y|}$$

where

$$|y|_{s=0} = g_{11}^2 - g_{12}^2. \quad (34)$$

Then

$$y'_{11}|_{s=0} = y'_{22}|_{s=0} = \frac{z'_{11}}{|z'|}|_{s=0} = g'_{11} = \frac{g_{11} - R|y|}{1 - 2Rg_{11} + |y|R^2} \quad (35)$$

$$y'_{12}|_{s=0} = -\frac{z'_{12}}{|z'|}|_{s=0} = g'_{12} = \frac{g_{12}}{1 - 2Rg_{11} + |y|R^2}$$

Consider the denominator of y'_{11} , y'_{12} , and y'_{22} . From (32) and (34),

$$1 - 2Rg_{11} + |y|R^2 = \frac{(\sum |k_{12i}|)^2 - g_{12}^2}{(\sum |k_{12i}| + g_{11})^2} \quad (36)$$

Since a_n is zero in (26),

$$\sum k_{12i} = -g_{12}$$

which requires that

$$\sum |k_{12i}| \geq |g_{12}|. \quad (37)$$

Therefore, the denominator in (35) is nonnegative, and Condition b, from (35), is

$$g_{11} - R|y| \geq -g_{12}. \quad (38)$$

The limiting condition is for (38) to be satisfied with the equal sign.

Using the equal sign, (38) can be written as

$$(g_{11} + g_{12})[1 - R(g_{11} - g_{12})] = 0. \quad (39)$$

The value of g_{11} equal to $(-g_{12})$ satisfies this equation. Using the value of R in (32), the second factor of (39) becomes

$$1 - R(g_{11} - g_{12}) = \frac{\sum |k_{12i}| + g_{12}}{\sum |k_{12i}| + g_{11}} \quad (40)$$

which does not give g_{11} a second root. However, it is desirable that this factor not be zero. Consider the denominator of (35).

$$1 - 2Rg_{11} + |y|R^2 = [1 - R(g_{11} - g_{12})][1 - R(g_{11} + g_{12})]. \quad (41)$$

The first factor of (41) is the same as the second factor of (39). If this factor is zero, i.e., if $\sum |k_{12i}|$ is equal to $(-g_{12})$, then y'_{11} ,

y'_{12} , and y'_{22} will have a pole at the origin, and Condition e will not be satisfied. In this instance, a resistance less than the maximum value should be extracted from z_{11} and z_{22} . Then the value of R in (39) will be smaller than that given in (32), and the value of this factor will not be zero.

Thus, from (39), g_{11} should be chosen equal to $(-g_{12})$ to satisfy Condition b. Then the maximum value of extractable resistance, from (32), is

$$R = \frac{1}{\sum |k_{121}| - g_{12}} \quad (42)$$

It is seen from (37) that R is always nonnegative. This method fails for $\sum |k_{121}|$ equal to g_{12} , since, for this case, y_{11} is equal to $(-y_{12})$ for all s , and $|y|$ is identically zero.

Condition c will be investigated in terms of the sign of g_{12} . Consider first that g_{12} is negative. Then g_{11} is positive. From (30), since R is positive, y_{11} is positive as $s \rightarrow \infty$. Then y_{11} has the general shape shown in Figure 22. Now z_{11} has the same zeros as y_{11} , and z_{11} is equal to R as $s \rightarrow \infty$. Since g_{11} is equal to $(-g_{12})$, $|y|$ in (33) is zero, and z_{11} has a pole at the origin. Therefore, z_{11} and z'_{11} have the general shapes shown in Figure 22. The zeros of z'_{11} are also the zeros of y'_{11} . Since both z'_{11} and z'_{12} are zero as $s \rightarrow \infty$, y'_{11} has a pole at $s = \infty$. Thus, y'_{11} has the general shape shown in Figure 22, and Condition c is satisfied.

If g_{12} is zero, y_{11} and y_{12} have a zero at the origin. Then $|y|$ has a second-ordered zero at the origin, causing z_{11} and

z_{12} to have a pole there. Therefore, z_{11} has the same general shape as for a negative g_{12} . From (35), since g_{12} and $|y|$ are both zero, y'_{11} has a zero at the origin. Thus, y'_{11} has the same shape as for g_{12} negative, except that the first zero of y'_{11} occurs at the origin, and Condition c is satisfied.

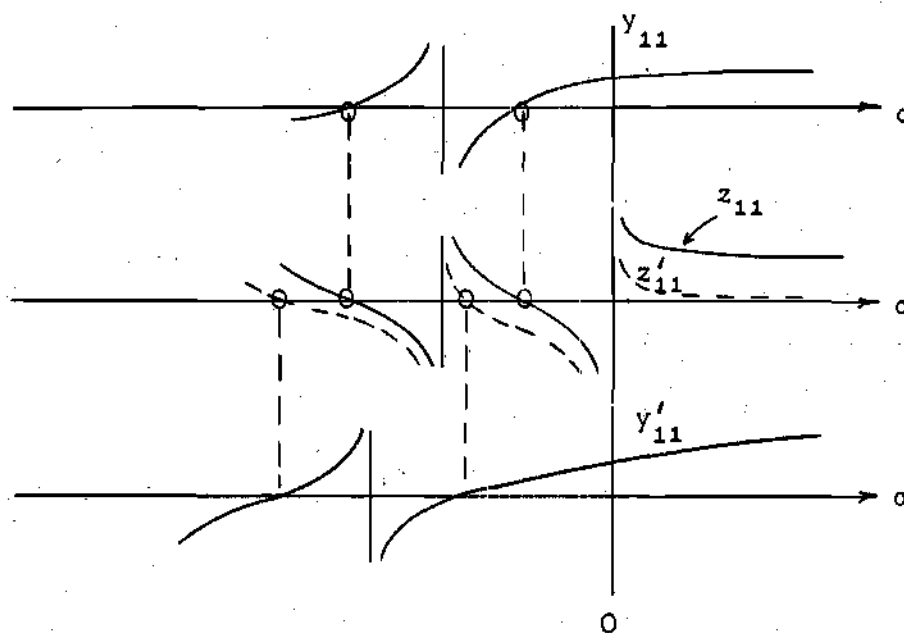


Figure 22. Immittance Functions for g_{12} Negative

If g_{12} is positive, g_{11} is negative. From (33), since R is positive, y_{11} is positive as $s \rightarrow \infty$, and y_{11} has the shape shown in Figure 23. By the same reasoning as for g_{12} negative, z_{11} and z'_{11} have the shapes shown in Figure 23, and y'_{11} has one zero on the positive real axis. Since y'_{11} has a pole at $s = \infty$, y'_{11} cannot have

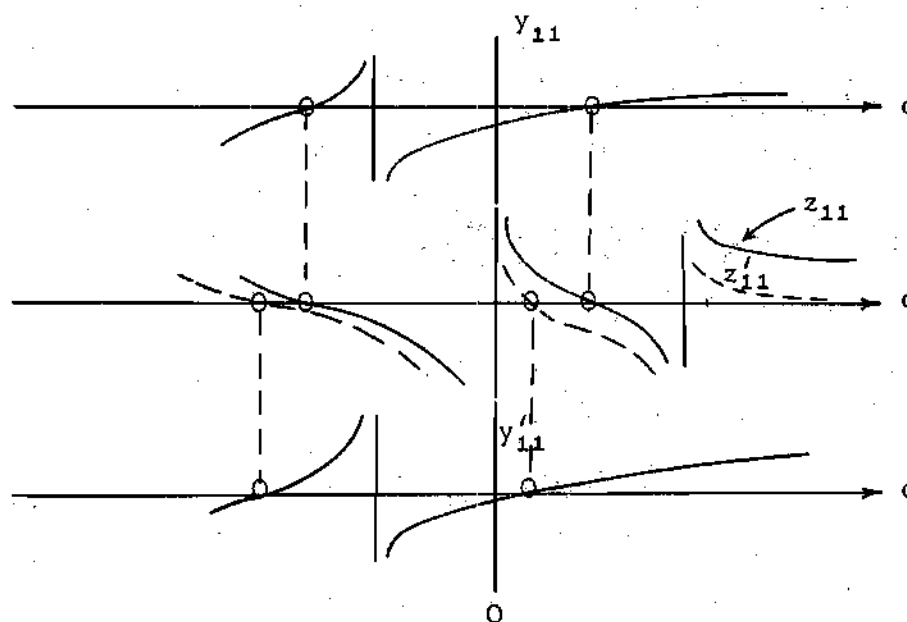


Figure 23, Immittance Functions for g_{12} Positive

a pole to the right of this zero. If g'_{11} is negative, y'_{11} cannot have a pole between the origin and this zero. From (35), since $|y|$ is zero and g_{11} is negative, g'_{11} is also negative. Therefore, Condition c is satisfied for all values of g_{12} .

Condition d is always satisfied unless $m = (n - 1)$. For this case, if a value of resistance less than R , of (42) is extracted from z_{11} and z_{22} , z'_{11} and z'_{22} will not be zero as $s \rightarrow \infty$, and $y'_{11}, y'_{12}, y'_{22}$ cannot have a pole there. It is seen from Figure 22 and Figure 23 that Condition c is still satisfied if y'_{11} is positive as $s \rightarrow \infty$.

As $s \rightarrow \infty$, z'_{12} is zero, and

$$y'_{11} = \frac{z'_{11}}{z'_{11} - z'_{12}} = \frac{1}{z'_{11}} \quad (43)$$

which is positive.

It is seen from the above discussion that y'_{11} , y'_{12} , and y'_{22} will have a pole at the origin only when $\sum |k_{12i}|$ is equal to $(-g_{12})$. This pole can be eliminated by extracting less than the maximum value of resistance from z_{11} and z_{22} . Thus, Condition e can always be satisfied.

In conclusion, for Case 1, the synthesis procedure will require only one negative resistor if R_1 and R_2 are each less than or equal to R , where

$$R = \frac{1}{\sum |k_{12i}| - g_{12}} ; \quad \sum |k_{12i}| \neq g_{12} \quad (44)$$

If $\sum |k_{12i}|$ is equal to g_{12} , the synthesis procedure fails. If $m = (n - 1)$, or if $\sum |k_{12i}|$ is equal to $(-g_{12})$, then R_1 and R_2 must be less than R . The synthesis procedure will require only two negative resistors if either R_1 or R_2 satisfies the above conditions, and the other does not.

Case 2

The admittance functions for Case 2 are the same as in (26), with $m = n$. Thus, $|y|$ has the same form as in (27). It is seen that for $m = n$,

$$b'_n = c_n d_n - a_n^2. \quad (45)$$

At $s = \infty$,

$$\begin{aligned} z_{11} \Big|_{s=\infty} &= \frac{y_{22}}{|y|} \Big|_{s=\infty} = \frac{d_n}{c_n d_n - a_n^2} = R_1 \\ z_{22} \Big|_{s=\infty} &= \frac{y_{11}}{|y|} \Big|_{s=\infty} = \frac{c_n}{c_n d_n - a_n^2} = R_2 \end{aligned} \quad (46)$$

where

$$\begin{aligned} c_n &= g_{11} + \sum k_{11i} \\ d_n &= g_{22} + \sum k_{22i} \end{aligned} \quad (47)$$

For Case 2, y'_{11} , y'_{12} , and y'_{22} can be expressed as in (31). The requirements for this set of functions to be realizable using only one negative resistor are then the same as in Case 1. To satisfy Condition a, k_{11i} and k_{22i} are both made equal to $|k_{12i}|$, g_{11} and g_{22} are made equal, and equal values of resistance are extracted from z_{11} and z_{22} .

The value of g_{11} is chosen such that Condition b is satisfied with an equal sign. The derivation of g'_{11} , g'_{12} , and g'_{22} is seen to be the same as in Case 1. With g'_{11} equal to $(-g'_{12})$, it is seen from (39) that

$$(g_{11} + g_{12})[1 - R(g_{11} - g_{12})] = 0. \quad (48)$$

There are two values of g_{11} that satisfy this equation. These values are

$$g_{11} = -g_{12} \quad \text{and} \quad g_{11} = \frac{a_n^2}{\sum |k_{12i}| + g_{12}} - \sum |k_{12i}|. \quad (49)$$

The second root is obtained by substituting (46) and (47) into the second factor of (48). As in Case 1, if the second factor of (48) is zero, y'_{11} , y'_{12} , and y'_{22} will have a pole at the origin. However, this difficulty can be overcome. The first root of (49) will be investigated now. From (46) and (47),

$$R = \frac{\sum |k_{12i}| - g_{12}}{(\sum |k_{12i}| - g_{12})^2 - a_n^2} \quad (50)$$

This value of R is nonnegative for three different conditions. These conditions are:

1. $g_{12} \leq 0$.
2. $g_{12} > 0$; $\sum |k_{12i}| - g_{12} > |a_n|$.
3. $g_{12} > 0$; $\sum |k_{12i}| - g_{12} < 0$; $|\sum |k_{12i}| - g_{12}| < |a_n|$.

In considering the circumstances for which Condition c is satisfied, each of the sets of conditions will be considered separately.

For Condition 1, since g_{12} is negative, g_{11} is positive. From (47), y_{11} is positive as $s \rightarrow \infty$, and y_{11} has the shape shown in Figure 24. Since $|y|$ in (34) is zero, z_{11} has a pole at the origin. As $s \rightarrow \infty$, z_{11} approaches R , and z_{11} and z'_{11} have the shapes shown in Figure 24. Since z'_{11} has a zero at $s = \infty$, and since z'_{12} does not have a zero there, y'_{11} will have a zero at $s = \infty$. In order that y'_{11} shall not have a pole on the positive real axis, it is necessary that g'_{11} be negative, as shown in Figure 24. From (35),

$$g'_{11} = \frac{g_{11}}{1 - 2Rg_{11}} \quad (51)$$

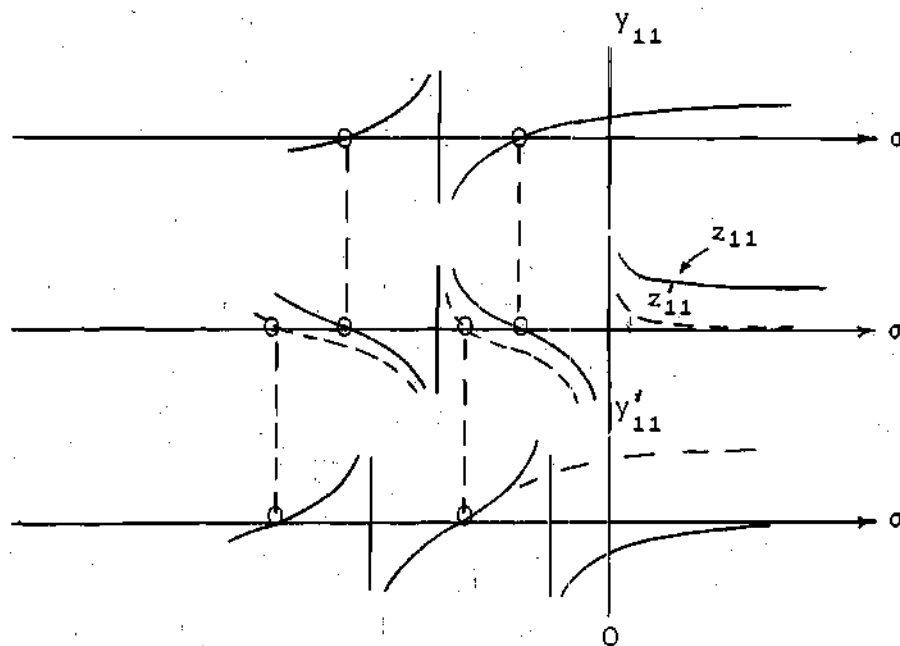


Figure 24. Immittance Functions for Condition 1

Since g_{11} is positive, the inequality

$$g_{11} > \frac{1}{2R} \quad (52)$$

must be satisfied for g'_{11} to be negative. If (50) and the second root of (49) is substituted into (52), it becomes

$$|g_{12}| > \frac{(\sum |k_{12i}| - g_{12})^2 - a_n^2}{2(\sum |k_{12i}| - g_{12})} \quad (53)$$

If this inequality is satisfied, y'_{11} will not have a pole on the positive real axis. If this inequality is not satisfied, y'_{11} can be

prevented from having a pole on the positive real axis by extracting a value of resistance less than R from z_{11} . To determine this smaller value of resistance, consider the sketch of y'_{11} . It is seen from the sketch that if y'_{11} is positive as $s \rightarrow \infty$, y'_{11} will not have a pole on the positive real axis. As $s \rightarrow \infty$,

$$y'_{11} = \frac{z'_{11}}{z'_{11} - z'_{12}} \quad (54)$$

Thus, for y'_{11} to be positive, it is necessary that z'_{11} be greater in magnitude than z'_{12} at infinity. From (46),

$$z'_{11} \Big|_{s=\infty} = z_{11} \Big|_{s=\infty} - R' = \frac{c_n}{c_n^2 - a_n^2} - R' \quad (55)$$

where R' is a value less than R . Also,

$$z'_{12} \Big|_{s=\infty} = z_{12} \Big|_{s=\infty} = \frac{a_n}{c_n^2 - a_n^2} \quad (56)$$

For z'_{11} to be greater in magnitude than z'_{12} at $s = \infty$,

$$\frac{c_n}{c_n^2 - a_n^2} - R' > \left| \frac{a_n}{c_n^2 - a_n^2} \right| = \frac{|a_n|}{c_n^2 - a_n^2} \quad (57)$$

Or,

$$R' < \frac{c_n - |a_n|}{c_n^2 - a_n^2} = \frac{1}{c_n + |a_n|} = \frac{1}{\sum |k_{12i}| - g_{12} + |a_n|} \quad (58)$$

It is seen that R' is always less than R . Therefore, if (53) is satisfied, the maximum value of extractable resistance is given by (50). If (53) is not satisfied, the maximum value of extractable resistance is given by (58).

If g_{11} is zero, y_{11} and y_{12} have a zero at the origin. Thus, $|y|$ has a second-ordered zero at the origin, causing z_{11} and z_{12} to have a pole there. Then z_{11} has the same general shape as for a negative g_{12} . From (35), since g_{12} and $|y|$ are both zero, y'_{11} has a zero at the origin. If all of the extractable resistance is removed from z_{11} and z_{22} , y'_{11} will have a pole on the positive real axis. Therefore, to prevent y'_{11} from having this pole, the extractable resistance is limited by (58).

For Condition 2, since g_{11} is negative and c_n is positive, y_{11} has the shape shown in Figure 25. From the discussion of Condition 1, z_{11} and z'_{11} have the shapes shown in Figure 25. It is seen that y'_{11} will always have a zero on the positive real axis. In order that there be no pole to the right of this zero, it is necessary that y'_{11} approach a positive value as s goes to infinity. In other words, from the discussion of Condition 1,

$$R' < \frac{1}{\sum |k_{121}| - g_{12} + |a_m|} \quad (59)$$

In order that no pole shall appear between the origin and this zero, it is necessary that g'_{11} be negative. From (51), since g_{11} is negative, g'_{11} is also negative. Thus, the limit on the extractable resistance for Condition 2 is given by (59).

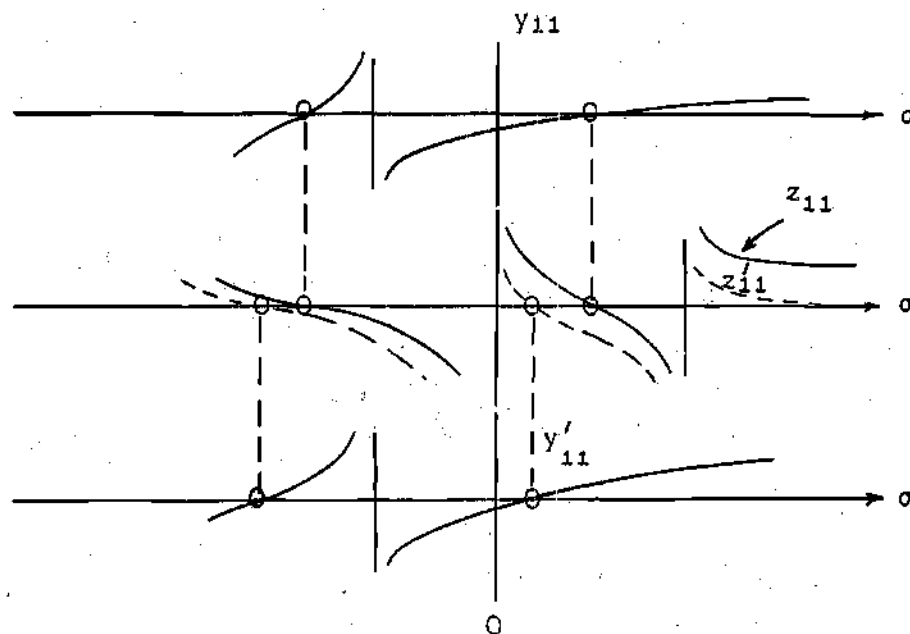


Figure 25. Immittance Functions for Condition 2

For Condition 3, since both g_{11} and c_n are negative, y_{11} has the shape shown in Figure 26. From the discussion of Condition 1, z_{11} and z'_{11} have the shapes shown in Figure 26. If g'_{11} is negative, it is seen that y'_{11} will not have a pole on the positive real axis. As was shown for Condition 2, g'_{11} is always negative for g_{11} negative. Thus, the maximum extractable resistance is given by (50).

Consider now the second root given in (49). If this value of g_{11} is used, and if a value of resistance less than the maximum value is extracted from z_{11} , the second factor of (48) does not go to zero and y'_{11} does not have a pole at the origin.

It will now be shown that, under certain conditions, the second root will allow more resistance to be extracted than the first root.

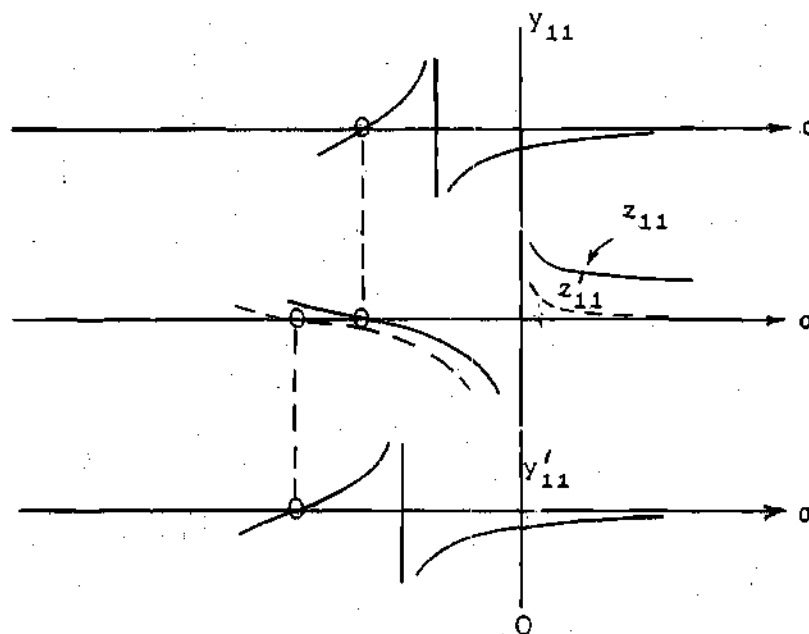


Figure 26. Immittance Functions for Condition 3

However, the number of negative resistors will never be fewer if the second root is used. By substituting the second root into (46) and (47), the extractable resistance

$$R_a = \frac{1}{g_{11} - g_{12}} = \frac{\sum |k_{12i}| + g_{12}}{a_n^2 - (\sum |k_{12i}| + g_{12})^2} \quad (60)$$

is obtained. Since

$$a_n = \sum k_{12i} + g_{12} \quad (61)$$

R_a is always nonpositive for g_{12} positive. For g_{12} negative and R_a positive, setting R_a greater than R of (50) yields the inequality

$$(\sum |k_{12i}|)^2 - (g_{12})^2 > a_n^2. \quad (62)$$

Thus, if g_{12} is negative and if (62) is satisfied, the extractable resistance is greater when the second root is used.

From (62),

$$\sum |k_{12i}| - g_{12} > \frac{a_n^2}{\sum |k_{12i}| + g_{12}}. \quad (63)$$

After substituting this inequality into the second root of (49), it is seen that g_{11} is less than $(-g_{12})$. Also, from (60), for R_a to be positive,

$$(\sum |k_{12i}| + g_{12})^2 < a_n^2. \quad (64)$$

Or,

$$\sum |k_{12i}| + g_{12} < \frac{a_n^2}{\sum |k_{12i}| + g_{12}}. \quad (65)$$

After substituting this inequality into the second root of (49), it is seen that g_{11} is greater than g_{12} . Thus,

$$g_{12} < g_{11} < -g_{12}. \quad (66)$$

Then $|y|$ of (34) is negative.

If the equality sign is used in (35), Condition b becomes

$$\frac{g_{11} - R|y|}{1 - 2Rg_{11} + |y| R^2} = \frac{-g_{12}}{1 - 2Rg_{11} + |y| R^2} \quad (67)$$

From (41), the denominator of (67) is

$$1 - 2Rg_{11} + |y| R^2 = [1 - R(g_{11} - g_{12})][1 - R(g_{11} + g_{12})]. \quad (68)$$

Since R is chosen smaller than the value that makes the first factor of the right side of (68) zero, this factor is positive. The second factor is also positive, since g_{12} is negative and always greater in magnitude than g_{11} . Thus, the denominator of (67) is positive. If (67) is multiplied through by its denominator, there results

$$g_{11} + |R| |y| = -g_{12}. \quad (69)$$

Now R must be smaller than the value that satisfies this equation, in order that y'_{11} will not have a pole at the origin. Then, for the second root, the left side of (69) is less than the right side. Since the denominator of (67) is positive, $g'_{11} < -g'_{12}$, Condition b is no longer satisfied, and two additional negative resistors are required to realize these conductance terms. For this reason, synthesis using the second root will never require fewer negative resistors than synthesis using the first root.

It is seen that for Conditions 1, 2, and 3, $k'_{12\infty}$ is zero, and Condition d of Case 1 is satisfied. It is also seen that y'_{11} , y'_{12} and y'_{22} never have a pole at the origin, and Condition e is satisfied.

In conclusion, for Case 2, if both R_1 and R_2 are less than the values of extractable resistance given for the various conditions, only one negative resistor is required in the synthesizing network.

If only one of the resistances is less, two negative resistors are required in the synthesizing network. A numerical example illustrating this case is given in Appendix III.

Case 3

For this case, since y_{12} has a pole at infinity, y_{11} and y_{22} must also have a pole there. The functions y'_{11} , y'_{12} , and y'_{22} are of the form given in (31). In order that only one negative resistor is required in the realization of these functions, the five conditions given in Case 1 must be satisfied.

To satisfy Condition a, k_{11i} and k_{22i} are both made equal to $|k_{12i}|$ in all poles, g_{11} and g_{22} are made equal, and equal values of resistance are extracted from z_{11} and z_{22} . Since k_{11i} and k_{22i} are both equal to $|k_{12i}|$ in all poles, y_{11} , y_{12} , and y_{22} can be expressed as

$$y_{11} = y_{22} = |k_{12\infty}|s + g_{11} + \sum \frac{|k_{12i}|s}{s + s_i} = \frac{c_0 + c_1s + \dots + c_{n+1}s^{n+1}}{b_0 + b_1s + \dots + s^n} \quad (70)$$

$$y_{12} = k_{12\infty}s + g_{12} + \sum \frac{k_{12i}s}{s + s_i} = \frac{a_0 + a_1s + \dots + a_{n+1}s^{n+1}}{b_0 + b_1s + \dots + s^n}$$

As in Cases 1 and 2, g_{11} is chosen to obtain maximum extractable resistance with Condition b satisfied. This value of g_{11} will now be found. Since the numerator of $|y|$ is of degree $(n+1)$ and the denominator is of degree n , the numerator and the denominator of z_{11} are both of degree $(n+1)$. Therefore, the value of resistance

extractable from z_{11} and z_{22} is the ratio of c_{n+1} and the coefficient of the highest ordered term in the numerator of $|y|$. From (70), this coefficient, call it b'_{n+1} , is seen to be

$$b'_{n+1} = 2g_{11} |k_{12\infty}| - 2g_{12} k_{12\infty} + 2 \sum k_{12a} (|k_{12\infty}| - k_{12\infty}) + 2 \sum k_{12b} (|k_{12\infty}| + k_{12\infty}) \quad (71)$$

where

$$\sum \frac{k_{12i}s}{s + s_i} = \sum \frac{k_{12a}s}{s + s_i} - \sum \frac{k_{12b}s}{s + s_i}, \quad k_{12a} > 0, \quad k_{12b} > 0. \quad (72)$$

The value of the extractable resistance is then

$$R = \frac{|k_{12\infty}|}{2g_{11} |k_{12\infty}| - 2g_{12} k_{12\infty} + 2 \sum k_{12a} (|k_{12\infty}| - k_{12\infty}) + 2 \sum k_{12b} (|k_{12\infty}| + k_{12\infty})} \quad (73)$$

For Condition b to be satisfied with an equal sign, from (39),

$$(g_{11} + g_{12})[1 - R(g_{11} - g_{12})] = 0. \quad (74)$$

By solving (73) and (74) simultaneously for g_{11} , two roots are obtained.

These roots are

$$g_{11} = -g_{12}$$

and

$$g_{11} = \frac{1}{|k_{12\infty}|} [g_{12}(2k_{12\infty} - |k_{12\infty}|) - 2 \sum k_{12a} (|k_{12\infty}| - k_{12\infty}) - 2 \sum k_{12b} (|k_{12\infty}| + k_{12\infty})]. \quad (75)$$

As in Cases 1 and 2, the second root will cause y'_{11} , y'_{12} , and y'_{22} to have a pole at the origin. This problem can be averted. However, the second root will never require fewer negative resistors than the first. Synthesis using the second root will be investigated after the consideration of the first root. When the first root is used, the maximum value of the extractable resistance is, from (73),

$$R = \frac{|k_{12\infty}|}{2(|k_{12\infty}| + k_{12\infty})(\sum k_{12b} - g_{12}) + 2 \sum k_{12a}(|k_{12\infty}| - k_{12\infty})}. \quad (76)$$

The resistance R is seen to be positive for two different conditions.

These conditions are

1. $g_{12} < 0$.
2. $g_{12} > 0$; and either $\sum k_{12a} > g_{12}$, or $k_{12\infty} < 0$, or both.

The circumstances under which Condition c is satisfied will be investigated in terms of these two conditions.

For Condition 1, g_{11} is positive. Since y_{11} has a pole at infinity, y_{11} has the shape shown in Figure 27. Since $|y|$ of (34) is zero, z_{11} has a pole at the origin, and has the shape shown in Figure 27. Since z'_{12} is not zero as $s \rightarrow \infty$, y'_{11} is zero. In order that y'_{11} will have no poles on the positive real axis, g'_{11} must be negative. From (51),

$$R > \frac{1}{2g_{11}} = \frac{1}{2|g_{12}|} \quad (77)$$

If this inequality is not satisfied, subtracting less resistance from z_{11} will not make y'_{11} positive at $s = \infty$, as in Case 2. At $s = \infty$,

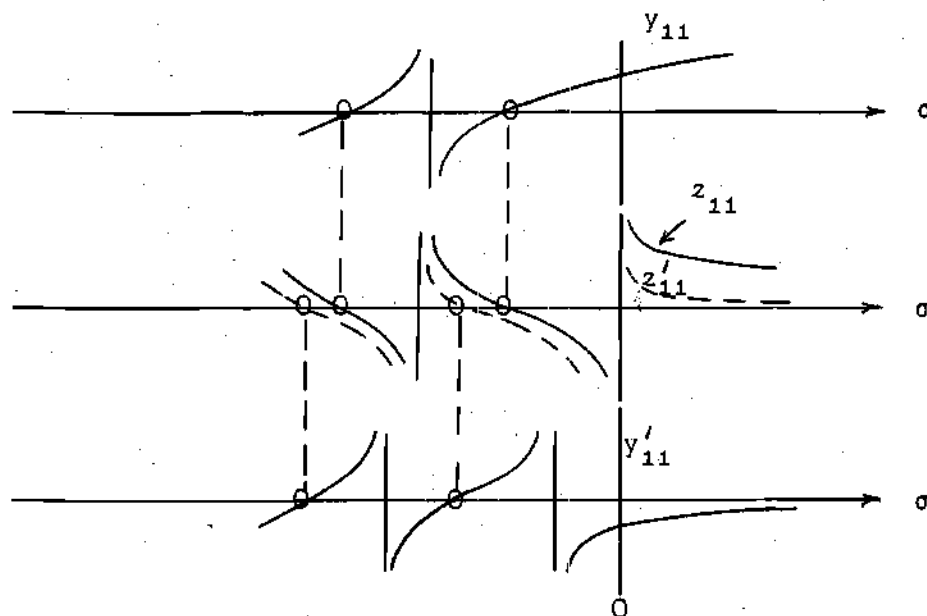


Figure 27. Immittance Functions for g_{12} Negative

z_{11} is equal to z_{12} . Therefore, unless (77) is satisfied, y'_{11} will have one pole on the positive real axis. This pole may be acceptable, since this pole will require only one or two additional negative resistors.

If g_{12} is zero, y_{11} and y_{12} have a zero at the origin, and $|y|$ has a second-ordered zero at the origin, causing z_{11} and z_{12} to have a pole there. Thus, z_{11} has the same general shape as for a negative g_{12} . From (35), since g_{11} and $|y|$ are both zero, y'_{11} has a zero at the origin, and, therefore, will always have a pole on the positive real axis. Once again, this pole may be acceptable.

For Condition 2, since g_{11} is negative, y_{11} has the shape shown in Figure 28. By the discussion of Condition 1, z_{11} , z'_{11} , and

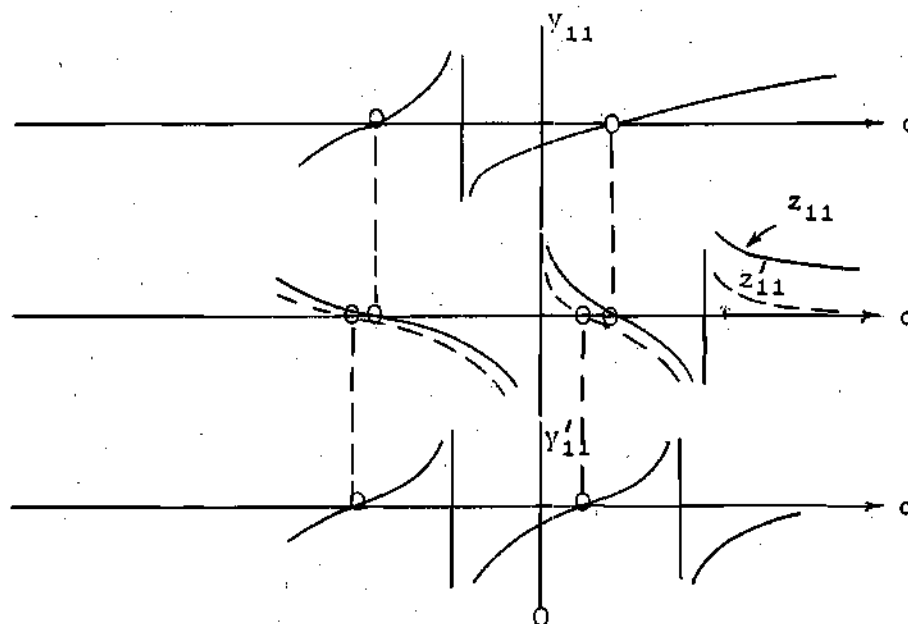


Figure 28. Immittance Functions for g_{12} Positive

y'_{11} have the shapes shown in Figure 28. It is seen that y'_{11} will have one pole on the positive real axis. Since, from (51), g'_{11} is negative, y'_{11} will have only one pole on the positive real axis. This pole may be acceptable, for the reasons given in the discussion of Condition 1.

Consider the second root for g_{11} , given in (75). When this root is used, the value of the extractable resistance is, from (73),

$$R_a = \frac{|k_{12\infty}|}{2(g_{12} + \sum k_{12a})(k_{12\infty} - |k_{12\infty}|) - 2 \sum k_{12b}(|k_{12\infty}| + k_{12\infty})} \quad (78)$$

By substituting R given by (76) into the inequality $R_a > R$, the following inequality is obtained.

$$\sum k_{12a}(|k_{12\infty}| - k_{12\infty}) + \sum k_{12b}(|k_{12\infty}| + k_{12\infty}) > k_{12\infty}g_{12}. \quad (79)$$

Consider four different cases in terms of $k_{12\infty}$ and g_{12} .

- a. $k_{12\infty} < 0, \quad g_{12} > 0.$
- b. $k_{12\infty} > 0, \quad g_{12} < 0.$
- c. $k_{12\infty} > 0, \quad g_{12} > 0.$
- d. $k_{12\infty} < 0, \quad g_{12} < 0.$

For the first three cases, R_a is negative. Thus, only the fourth case need be considered. For Case d, R_a is positive if $|g_{12}|$ is greater than $\sum k_{12a}$. Equation (79) is satisfied if $2 \sum k_{12a}$ is greater than $|g_{12}|$. Thus, for R_a to be positive and greater than R ,

$$\frac{|g_{12}|}{2} < \sum k_{12a} < |g_{12}|, \quad k_{12\infty} < 0, \quad g_{12} < 0. \quad (80)$$

With $k_{12\infty}$ negative, the second root in (75) becomes

$$g_{11} = 3|g_{12}| - 4 \sum k_{12a}. \quad (81)$$

It is seen from (80) and (81) that

$$-|g_{12}| < g_{11} < |g_{12}| \quad (82)$$

As in Case 2, the second root causes $|y|$ of (34) to be negative. For the same reasons given in Case 2, the denominator of (67) is positive. Thus, by the discussion of Case 2, additional negative resistors are required to realize the conductance terms of y'_{11} , y'_{12} , and y'_{22} . For this reason, the second root will never require fewer negative resistors than the first.

It is seen that for Conditions 1 and 2, $k'_{12\infty}$ is always zero, and Condition d of Case 1 is satisfied. It is also seen that y'_{11} , y'_{12} , and y'_{22} never have a pole at the origin, and Condition e is satisfied.

Thus, in many cases, y'_{11} will have a pole on the real positive axis. However, this pole may be acceptable, since synthesis by the direct method will always require at least three negative resistors, and, if k_{12} is positive, the direct method will require five resistors.

APPENDICES

APPENDIX I

INTERNAL NEGATIVE RESISTOR REQUIREMENTS
FOR VARIOUS POLES

For poles not on the finite negative real axis, internal negative resistors are required in the synthesizing networks given in Chapter II. It will be shown here that internal negative resistors are necessary for these poles.

Consider Figure 29. Let y_{11}' , y_{12}' , and y_{22}' be the admittance functions of the total network. If all negative resistors are to be in the desirable locations, y_{11}' , y_{12}' , and y_{22}' must be RC admittance

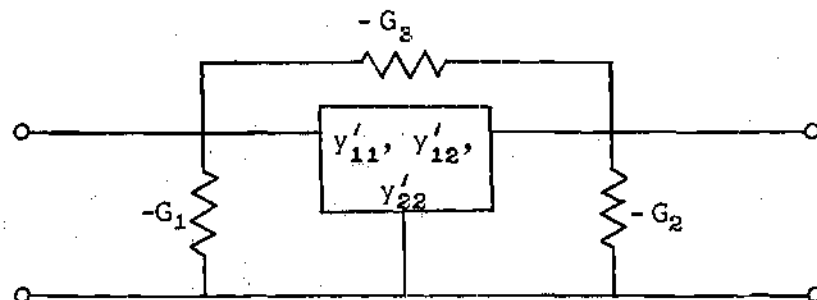


Figure 29. Desirable Locations for Negative Resistors

functions. Now,

$$y_{11} = y_{11}' - (G_1 + G_3)$$

$$y_{12} = y_{12}' + G_3$$

$$y_{22} = y_{22}' - (G_2 + G_3) .$$

Or,

$$Y'_{11} = Y_{11} + G_1 + G_3$$

$$Y'_{12} = Y_{12} - G_3$$

$$Y'_{22} = Y_{22} + G_2 + G_3$$

If the overall network is to realize a pole at infinity, then

$$Y'_{11} = k_{11}s + G_1 + G_3$$

$$Y'_{12} = k_{12}s - G_3$$

$$Y'_{22} = k_{22}s + G_2 + G_3$$

This set of admittance functions is not realizable as an RC network unless the pole itself is realizable as an RC network. Thus, in general, internal negative resistors are required for the pole at infinity.

If the overall network is to realize a pole at the origin, then

$$Y'_{11} = \frac{k_{11}}{s} + G_1 + G_3$$

$$Y'_{12} = \frac{k_{12}}{s} - G_3$$

$$Y'_{22} = \frac{k_{22}}{s} + G_2 + G_3$$

This set of admittance functions cannot be realized by an RC network.

If the overall network is to realize a pole on the positive real axis, then

$$y'_{11} = \frac{k_{11}s}{s + s_1} + G_1 + G_3 = \frac{(k_{11} + G_1 + G_3)s + (G_1 + G_3)s_1}{s + s_1}$$

$$y'_{12} = \frac{k_{12}s}{s + s_1} - G_3 = \frac{(k_{12} - G_3)s - G_3s_1}{s + s_1}$$

$$y'_{22} = \frac{k_{22}s}{s + s_1} + G_2 + G_3 = \frac{(k_{22} + G_2 + G_3)s + (G_2 + G_3)s_1}{s + s_1}$$

Since s_1 is negative, this set of admittance functions cannot be realized by an RC network.

APPENDIX II

COMBINATIONS OF POLES NOT LEADING TO A REDUCTION IN
THE NUMBER OF REQUIRED NEGATIVE RESISTORS

Let the admittance functions of the two internal poles be expressed as

$$y_{11} = \frac{k_{11a}s}{s + s_a} + \frac{k_{11b}s}{s + s_b} = \frac{s^2(k_{11a} + k_{11b}) + s(k_{11a}s_b + k_{11b}s_a)}{s^2 + s(s_a + s_b) + s_a s_b}$$

$$y_{12} = \frac{k_{12a}s}{s + s_a} + \frac{k_{12b}s}{s + s_b} = \frac{s^2(k_{12a} + k_{12b}) + s(k_{12a}s_b + k_{12b}s_a)}{s^2 + s(s_a + s_b) + s_a s_b}$$

$$y_{22} = \frac{k_{22a}s}{s + s_a} + \frac{k_{22b}s}{s + s_b} = \frac{s^2(k_{22a} + k_{22b}) + s(k_{22a}s_b + k_{22b}s_a)}{s^2 + s(s_a + s_b) + s_a s_b}$$

Consider first the case of $s_a > 0$, $s_b > 0$, $k_{12a} > 0$, and $k_{12b} > 0$. The coefficient of the capacitive term in the equivalent z_{12} is negative [see (7), Chapter III], and this pair of poles is not realizable by the method of Chapter III.

Consider next the case of $s_a < 0$, $s_b > 0$, $k_{12a} < 0$, and $k_{12b} > 0$. The coefficient of the capacitive term in the equivalent z_{12} is negative, and this pair of poles is not realizable by the method of Chapter III.

Consider next the case for $s_a < 0$, $s_b < 0$, $k_{12a} < 0$, and $k_{12b} < 0$. The coefficient of the capacitive term in the equivalent z_{12} is negative, and this pair of poles is not realizable by the method of Chapter III.

Consider the combination of a pole at the origin and a pole on the finite real axis.

$$y_{11} = \frac{k_{11a}}{s} + \frac{k_{11b}s}{s + s_b} = \frac{k_{11b}s^2 + k_{11a}s + k_{11a}s_b}{s(s + s_b)}$$

$$y_{12} = \frac{k_{12a}}{s} + \frac{k_{12b}s}{s + s_b} = \frac{k_{12b}s^2 + k_{12a}s + k_{12a}s_b}{s(s + s_b)}$$

$$y_{22} = \frac{k_{22a}}{s} + \frac{k_{22b}s}{s + s_b} = \frac{k_{22b}s^2 + k_{22a}s + k_{22a}s_b}{s(s + s_b)}$$

Then

$$z_{11} = \frac{y_{22}}{|y|} = \frac{1}{k} (k_{22b}s + k_{22a} + \frac{k_{22a}s_b}{s})$$

$$z_{12} = -\frac{y_{12}}{|y|} = -\frac{1}{k} (k_{12b}s + k_{12a} + \frac{k_{12a}s_b}{s})$$

$$z_{22} = \frac{y_{11}}{|y|} = \frac{1}{k} (k_{11b}s + k_{11a} + \frac{k_{11a}s_b}{s})$$

For this set of impedances to be realizable with the poles compact, it is necessary that $k_{11a} = k_{22a} = k_{12a}$. In general, this will not be true. Also, no method is known for realizing the pole at infinity unless the same type of relationship is true for pole b. Thus, there is no advantage in this combination.

It is apparent that like conditions exist for the combination of a pole at infinity with a finite pole, or for the combination of a pole at infinity with a pole at the origin. The basic synthesis technique itself is a combination of conductance terms with the various types of poles.

APPENDIX III

NUMERICAL EXAMPLES

Chapter III, Case A

Consider the short-circuit admittance functions

$$y_{11} = \frac{100s}{s+1} + \frac{s}{s+2}$$

$$y_{12} = \frac{-10s}{s+1} + \frac{-2s}{s+2}$$

$$y_{22} = \frac{s}{s+1} + \frac{4s}{s+2}$$

The synthesis of these functions by the direct application of the methods of Chapter II requires two negative resistors. Equations (9) of Chapter III are not satisfied, but substitution into (10) yields

$$\frac{2}{9} \geq r \geq \frac{1}{90}$$

$$\frac{1}{9} \geq r \geq \frac{1}{180}$$

By letting $r = (1/9)$, the admittance functions can be expressed as

$$y_{11} = \frac{\frac{100}{9}s}{s+1} + \frac{s}{s+2} + \frac{\frac{800}{9}s}{s+1}$$

$$y_{12} = \frac{-\frac{10}{9}s}{s+1} + \frac{-2s}{s+2} + \frac{-\frac{80}{9}s}{s+1}$$

$$y_{22} = \frac{\frac{1}{9}s}{s+1} + \frac{4s}{s+2} + \frac{\frac{8}{9}s}{s+1}$$

Transformation of the first two poles of these functions to the equivalent open-circuit impedances of (7) yields

$$z_{11} = \frac{37}{361} + \frac{38}{361s}$$

$$z_{12} = \frac{28}{361} + \frac{38}{361s}$$

$$z_{22} = \frac{109}{361} + \frac{209}{361s}$$

The network for these impedance functions is given in Figure 30a. The

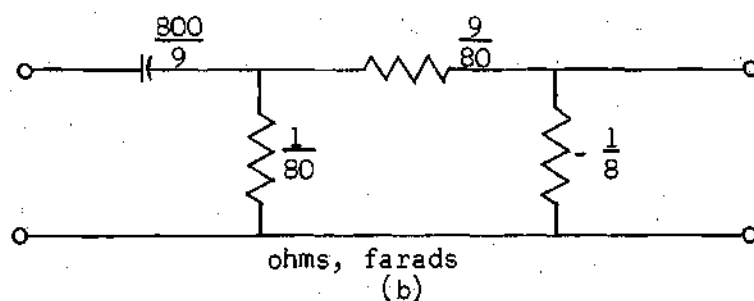
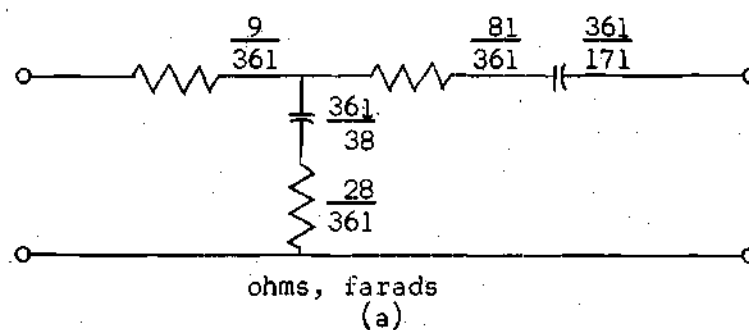


Figure 30. Networks for Example of Case A, Chapter III.

network for the remainder of the pole at $s = (-1)$ is given in Figure 30b. The network for the total admittance functions is thus the parallel connection of these two networks, and only one negative resistor is required.

Chapter IV, Case C

Consider the short-circuit admittance functions

$$y_{11} = \frac{-6s^2 + 9s - 2}{s^2 - 3s + 2} = \frac{-s}{s-1} + \frac{-4s}{s-2} - 1$$

$$y_{12} = \frac{-2s}{s^2 - 3s + 2} = \frac{2s}{s-1} + \frac{-2s}{s-2}$$

$$y_{22} = \frac{-6s^2 + 12s - 2}{s^2 - 3s + 2} = \frac{-4s}{s-1} + \frac{-s}{s-2} - 1.$$

If $G_S = 2$, and $G_L = 3$, then

$$G_S + \frac{G_L y_{11} + y_{11} y_{22} - y_{12}^2}{y_{22} + G_L} = \frac{12s^2 + 15s + 4}{-3s^2 + 3s + 4}.$$

All of the zeros of this admittance fall on the negative real axis, and the resulting network is stable.

Chapter IV, Case D

Consider the short-circuit admittance functions

$$y_{11} = \frac{-6s - 6}{s^2 + s - 2} = \frac{-4s}{s-1} + \frac{s}{s+2} + 3$$

$$y_{12} = \frac{6s}{s^2 + s - 2} = \frac{2s}{s - 1} + \frac{-2s}{s + 2}$$

$$y_{22} = \frac{2s^2 - 7s + 2}{s^2 + s - 2} = \frac{-s}{s - 1} + \frac{4s}{s + 2} - 1.$$

If $G_L = 2$,

$$z_{22} + \frac{1}{G_L} = \frac{y_{11}}{y_{11}y_{22} - y_{12}^2} + \frac{1}{G_L} = \frac{12s + 3}{12s - 6}.$$

All of the zeros of this impedance fall on the negative real axis, and the resulting network is stable.

Chapter IV, Case E

Consider the short-circuit admittance functions

$$y_{11} = \frac{-4s^2 + 3s + 2}{s^2 - 3s + 2} = \frac{-s}{s - 1} + \frac{-4s}{s - 2} + 1$$

$$y_{12} = \frac{-4s}{s^2 - 3s + 2} = \frac{2s}{s - 1} + \frac{-2s}{s - 2}$$

$$y_{22} = \frac{-4s^2 + 6s + 2}{s^2 - 3s + 2} = \frac{-4s}{s - 1} + \frac{-s}{s - 2} + 1$$

Then

$$Y_i = \frac{y_{11}y_{22} - y_{12}^2}{y_{22}} = \frac{16s^2 + 12s + 2}{-4s^2 + 6s + 2}.$$

All of the zeros of this admittance fall on the negative real axis, and the resulting network is stable.

Chapter V

The function

$$Y_{12} = \frac{p(s)}{q(s)} = \frac{-(s^2 - 2s + 1)}{s^2 + 8s + 15} = \frac{-(s - 1)^2}{(s + 3)(s + 5)}$$

is to be synthesized. Now,

$$k_{12g} = \frac{a_0}{k \prod s_i} = \frac{-1}{k \prod s_i}$$

$$k_{22g} = \frac{b_0 - k \prod s_i}{k \prod s_i} = \frac{15 - k \prod s_i}{k \prod s_i}$$

In the region $(-3) \leq s \leq (-5)$, $p(s)/q(s)$ is positive. Thus, a negative resistor is required across the top terminals of the network. Since $a_m = (-1)$, and since k can be chosen positive, $q_2(s)$ can be chosen to have only one zero. If s_1 is chosen as 4, and k as $7/2$, no negative resistor is required for the conductance terms, and only one negative resistor is required for the total network. Thus, $q_2(s) = (7/2)(s + 4)$ and

$$Y_{12} = \frac{p(s)}{q_2(s)} = \frac{-(s^2 - 2s + 1)}{\frac{7}{2}(s + 4)} = -\frac{2}{7}s - \frac{1}{14} + \frac{\frac{25}{14}s}{s + 4}$$

$$Y_{22} = \frac{q(s) - q_2(s)}{q_2(s)} = \frac{s^2 + (9/2)s + 1}{\frac{7}{2}(s + 4)} = \frac{2}{7}s + \frac{1}{14} + \frac{\frac{1}{14}s}{s + 4}$$

The network for the pole at infinity is a capacitor of $2/7$ farad across the top terminals of the network, and the network for the

conductance terms is a resistance of 14 ohms across the top terminals of the network. The network for the pole at $s = (-4)$ is shown in Figure 31a, and the total network is shown in Figure 31b.

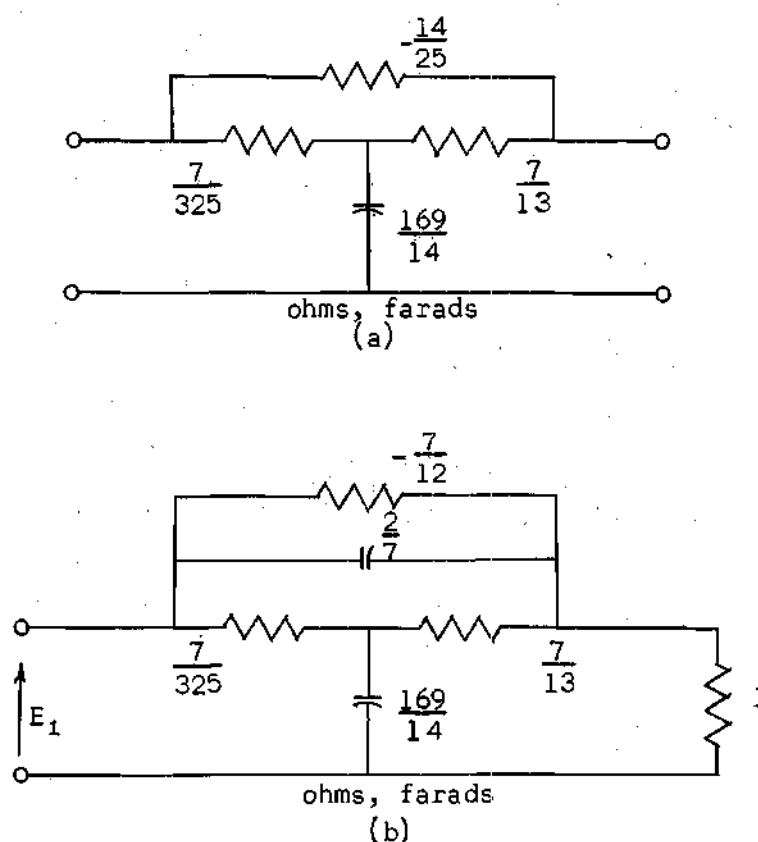


Figure 31. Networks for Example of Chapter V

Chapter VI

The function

$$\frac{E_2(s)}{E_1(s)} = \frac{(s - 1)}{s^2 + 6s + 5} = \frac{(s - 1)}{(s + 1)(s + 5)}$$

is to be synthesized. Now,

$$k_{12g} = \frac{-a_0}{k \prod s_i} = \frac{1}{k \prod s_i}$$

$$k_{22g} = \frac{b_0}{k \prod s_i} = \frac{5}{k \prod s_i}$$

Now, k can be chosen positive; thus, $R(s)$ should have only one zero, and this zero must occur between $(-1) \leq s \leq (-5)$. In this region, $P(s)/Q(s)$ is negative, and at $s = (-2)$, the magnitude is equal to one. Letting $k = 1$, $R(s) = (s + 2)$, and

$$Y_{12} = \frac{-(s - 1)}{s + 2} = \frac{1}{2} + \frac{-\frac{3}{2}s}{s + 2}$$

$$Y_{22} = \frac{s^2 + 6s + 5}{s + 2} = s + \frac{5}{2} + \frac{\frac{3}{2}s}{s + 2}$$

The required network is shown in Figure 32.

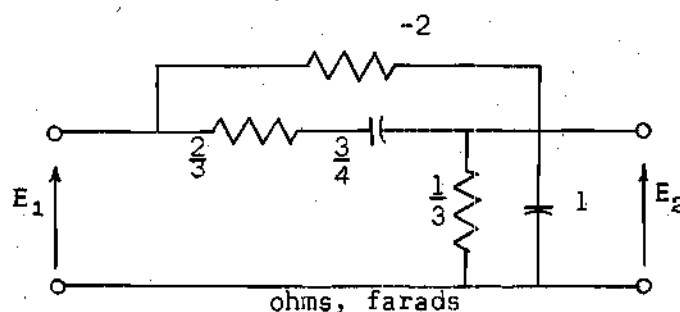


Figure 32. Network for Example of Chapter VI.

Chapter VII, Case 2

The function

$$\frac{E_2(s)}{E_1(s)} = \frac{\frac{1}{8}(s^2 - s)}{s^2 + 3s + 2}$$

is to be synthesized, with $R_1 = R_2 = (1/8)$ ohm. From (23),

$$y_{12} = \frac{-s^2 + s}{s^2 + 3s + 2} = \frac{2s}{s+1} + \frac{-3s}{s+2}$$

Since (53) is not satisfied, the maximum value of resistance is given by (58), From (58), $R' < (1/6)$ ohm. Therefore, only one negative resistor is required. Now, $g_{11} = (-g_{12}) = 0$, and

$$y_{11} = y_{22} = \frac{2s}{s+1} + \frac{3s}{s+2} = \frac{5s^2 + 7s}{s^2 + 3s + 2}$$

Thus,

$$z_{11} = z_{22} = \frac{y_{11}}{y_{11}^2 - y_{12}^2} = \frac{5s + 7}{24s} = \frac{1}{8} + \frac{2s + 7}{24s} = R_1 + z'_{11}$$

$$z_{12} = z'_{12} = \frac{-y_{12}}{y_{11}^2 - y_{12}^2} = \frac{s - 1}{24s}$$

Therefore,

$$y'_{11} = y'_{22} = \frac{z'_{11}}{z'_{11} z'_{12}} = \frac{s(16s + 56)}{s^2 + 10s + 16} = \frac{4s}{s+2} + \frac{12s}{s+8}$$

$$y'_{12} = \frac{-z'_{12}}{z'_{11} z'_{12}} = \frac{-8s(s-1)}{s^2 + 10s + 16} = \frac{-4s}{s+2} + \frac{12s}{s+8}$$

The network for the pole at $s = (-2)$ is shown in Figure 33a, the network for the pole at $s = (-8)$ is shown in Figure 33b, and the total network is shown in Figure 33c.

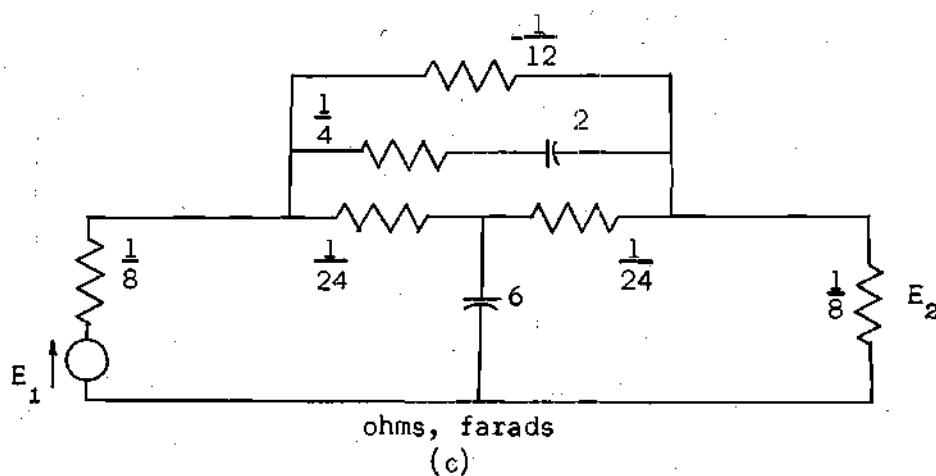
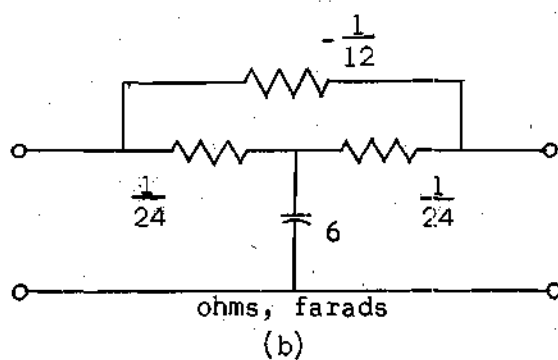
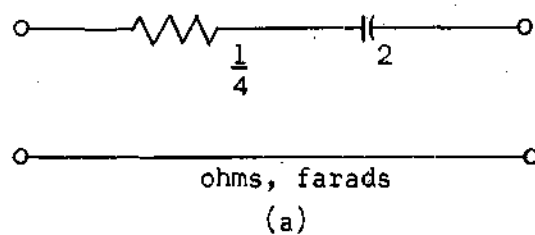


Figure 33. Networks for Example of Case 2, Chapter VII

BIBLIOGRAPHY

- (1) Su, Kendall L., "Active Network Synthesis," Preprint of a technical note for Project A-397, Engineering Experiment Station, Georgia Institute of Technology, October, 1961. Also E. E. 764 class notes.
- (2) Carlin, H. J., and D. C. Youla, "Network Synthesis with Negative Resistors," Proceedings of the Institute of Radio Engineers, 49, May, 1961, p. 907.
- (3) Youla, D. C., and L. I. Smilen, "Optimum Negative-Resistance Amplifiers," Proceedings of Symposium on Active Networks and Feedback Systems, Microwave Res. Inst. Symposia Series, Poly. Inst. Brooklyn, Vol. 10, 1960, p. 241.
- (4) Sard, E. W., "Gain-Bandwidth Performance of Maximally Flat Negative-Conductance Amplifiers," Proceedings of Symposium on Active Networks and Feedback Systems, Microwave Res. Inst. Symposia Series, Poly. Inst. Brooklyn, Vol. 10, 1960, p. 319.
- (5) Kawakami, M., T. Yanagisawa, and H. Shibayama, "Highly Selective Bandpass Filters Using Negative Resistances," Proceedings of Symposium on Active Networks and Feedback Systems, Microwave Res. Inst. Symposia Series, Poly. Inst. Brooklyn, Vol. 10, 1960, p. 369.
- (6) Kinariwala, B. K., "The Esaki Diode as a Network Element," IRE Trans. on Circuit Theory, CT-8, December, 1961, p. 389.
- (7) Weinberg, L., "Synthesis Using Tunnel Diodes and Masers," IRE Trans. on Circuit Theory, CT-8, March, 1961, p. 66.
- (8) Kinariwala, B. K., "Necessary and Sufficient Conditions for the Existence of $\pm R, C$ Networks," IRE Trans. on Circuit Theory, CT-7, September, 1960, p. 330.
- (9) Bello, P., "Extension of Brune's Energy Function Approach to the Study of LLF Networks," IRE Trans. on Circuit Theory, CT-7, September, 1960, p. 270.
- (10) Su, Kendall L., "Cascade Synthesis of RC Networks Using Negative Resistances," to be published in IRE Trans. on Circuit Theory, CT-9, December, 1962.

- (11) Lucal, H. M., "Synthesis of Three-Terminal RC Networks," IRE Trans. on Circuit Theory, CT-2, December, 1955, p. 308.
- (12) Kuh, E. S., "Synthesis of RC Grounded Two-Ports," IRE Trans. on Circuit Theory, CT-5, March, 1958, p. 55.

VITA

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